Entanglement Transformations

(Dolaşıklık Dönüşümleri)

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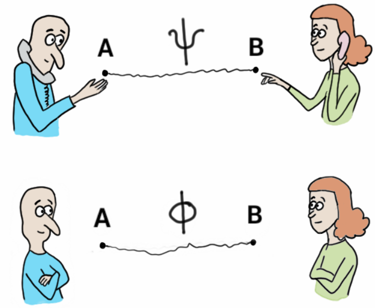
Entanglement Transformations

- What is it and Why study it?
- Bipartite pure
  - Asymptotic
  - Single Copy
  - Catalysis
- Multipartite
  - W-type states
  - Rank 2 states



- Ali is in Ankara with particle A
- Berna is in Bursa with particle B
- A and B are entangled with state $|\Psi\rangle_{AB}$
- Can Ali & Berna change the state to $|\Phi\rangle_{AB}$?

LOCC Local Operations with Classical Communication:

- Local unitary transformations (LU), i.e., time-evolution with local Hamiltonians.
- Local measurements.
- Sharing the measurement outcomes with classical communication.

Why do we need "entanglement transformation"?

Really, why?

- Entanglement is a useful resource.
  - **Teleportation:** CH Bennett *et al.*, PRL **70**, 1895 (1993).
  - **Dense Coding:** CH Bennett, SJ Wiesner, PRL **69**, 2881 (1992).
- But, maximally entangled states are used in teleportation and dense coding. RF Werner, *All teleportation and dense coding schemes*, J Phys A: Math Gen **34**, 7081 (2001).

Maximally entangled states: Same Schmidt coefficient.

$$\begin{aligned}
|\Psi_2\rangle_{AB} &= \frac{1}{\sqrt{2}}\Big(|1_A 1_B\rangle + |2_A 2_B\rangle\Big) \\
|\Psi_3\rangle_{AB} &= \frac{1}{\sqrt{3}}\Big(|1_A 1_B\rangle + |2_A 2_B\rangle + |3_A 3_B\rangle\Big) \\
&\vdots \qquad\qquad \vdots \\
|\Psi_d\rangle_{AB} &= \frac{1}{\sqrt{d}}\Big(|1_A 1_B\rangle + |2_A 2_B\rangle + \cdots + |d_A d_B\rangle\Big)
\end{aligned}$$

Entanglement can be quantified and classified by using the transformations

- Entanglement cannot be created out of nothing (by LOCC).
  - Product state: $|\Psi\rangle_{AB} = |\alpha_A \otimes \beta_B\rangle$.
  - Separable state: $\rho_{AB} = \sum p_i \left|\alpha_A^i \otimes \beta_B^i\right\rangle \langle \alpha_A^i \otimes \beta_B^i|$
- **quantum channels** must be used to establish entanglement.
- LOCC transformations enable us to *quantify* the entanglement contained in a state.
- **Entanglement Monotones:** Quantities that do not increase *on the average* during entanglement transformations. (G Vidal, J Mod Opt **47**, 355 (2000))

Partial Order on Entanglement

If $|\Psi\rangle \xrightarrow[LOCC]{} |\Phi\rangle$, then $|\Psi\rangle$ is more entangled than $|\Phi\rangle$.

LU equivalence

- Can we revert a transformation?

$$|\Psi\rangle_{AB\cdots D} \xrightarrow[LOCC]{} |\Phi\rangle_{AB\cdots D} \xrightarrow[LOCC]{} |\Psi\rangle_{AB\cdots D}$$

This can happen $\iff$ the states are related by local unitary (LU) transformations.

$$|\Psi\rangle_{AB\cdots D} = (U_A \otimes \cdots \otimes V_D)\,|\Phi\rangle_{AB\cdots D} \qquad U_A, \ldots, V_D : \text{unitary}$$

CH Bennett *et al.* PRA **63**, 012307 (2000).

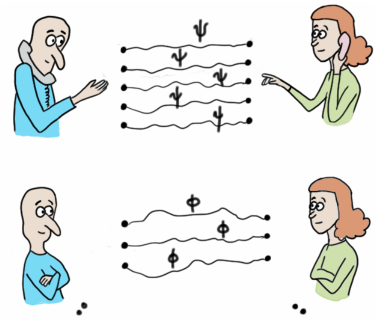
- Entanglement transformation is irreversible $\Leftrightarrow$ local measurements are used
- Transformation can be considered to be between LU-equivalence classes
- Bipartite pure states: LU-equivalent $\iff$ same $(x_1, \ldots, x_d)$ (Schmidt coef).

$$|\Psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{x_i}\,\left|\alpha_A^i \otimes \beta_B^i\right\rangle \quad \xleftrightarrow[LU]{} \quad |\Phi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{x_i}\,\left|u_A^i \otimes v_B^i\right\rangle$$

Types

Various types of entanglement transformations

- **Asymptotic:** Many copies states are transformed with sufficiently high fidelity. (e.g., convert 100,000 copies of state $|\Psi\rangle$ to 30,000 copies of $|\Phi\rangle$ with high fidelity.)
- **Single Copy:** Only a single pair of particles are in state $|\Psi\rangle$. Do something and change the state to $|\Phi\rangle$ exactly.
- **Deterministic vs Probabilistic:** Do you need a single final state with 100% probability or many possible final states with various probabilities.
- **Purification:** You have a mixed state. Try to change this to a nearly pure state.
- **Multipartite:** 3 or more persons at different locations.



$$\left|\Psi^{\otimes N}\right\rangle \xrightarrow[LOCC]{} \left|\Phi^{\otimes M}\right\rangle$$

with sufficiently high fidelity.

Entanglement entropy

$$|\Psi\rangle_{AB} = \sqrt{x_1}\,|1_A 1_B\rangle + \cdots + \sqrt{x_d}\,|d_A d_B\rangle$$
$$E(\Psi) = S(\rho_\Psi^A) = S(\rho_\Psi^B) = \sum_i -x_i \log_2 x_i$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \longrightarrow E(\Psi) = 1.$$

Asymptotic bipartite pure

$\left|\Psi^{\otimes N}\right\rangle \xrightarrow[LOCC]{} \left|\Phi^{\otimes M}\right\rangle$ is possible $\iff$ $NE(\Psi) \gtrsim ME(\Phi)$.

CH Bennett *et al.*, PRA **53**, 2046 (1996)

- Math is like Shannon's derivation of information entropy!

$$
\begin{aligned}
|\Psi\rangle &= \sum_i \sqrt{x_i}\, |i_A i_B\rangle \\
\left|\Psi^{\otimes N}\right\rangle &\approx \cdots + \underbrace{2^{-NE(\Psi)/2}\, |i_1 \cdots i_N\rangle_A \otimes |i_1 \cdots i_N\rangle_B}_{\text{“typical term”}} + \cdots \\
&\approx \frac{1}{2^{NE(\Psi)/2}} \sum_{I=1}^{2^{NE(\Psi)}} |I\rangle_A \otimes |I\rangle_B \\
&= 2^{NE(\Psi)} \text{ dimensional maximally entangled state} \\
&\approx \left|\Phi^{\otimes M}\right\rangle \qquad (\text{if } NE(\Psi) \approx ME(\Phi)).
\end{aligned}
$$

- Only a single pair AB in state $|\Psi\rangle_{AB}$.
- Can Ali & Berna change the state to $|\Phi\rangle_{AB}$?

$$|\Psi\rangle_{AB} = \sum_{i=1}^{n} \sqrt{x_i}\, |i_A i_B\rangle$$

$$|\Phi\rangle_{AB} = \sum_{i=1}^{n} \sqrt{y_i}\, |i_A i_B\rangle$$

$$\mathbf{x} = (x_1, x_2, \cdots, x_n) \equiv \lambda_\Psi = \lambda(\rho_\Psi^A) = \lambda(\rho_\Psi^B)$$

$$\mathbf{y} = (y_1, y_2, \cdots, y_n) \equiv \lambda_\Phi = \lambda(\rho_\Phi^A) = \lambda(\rho_\Phi^B)$$

Nielsen - Single Copy

$$|\Psi\rangle \xrightarrow[LOCC]{} |\Phi\rangle \quad \Longleftrightarrow \quad \lambda_\Psi \prec \lambda_\Phi$$

MA Nielsen, PRL **83**, 436 (1999).

Majorization

$\mathbf{x}$ is majorized by $\mathbf{y}$, or $\mathbf{x} \prec \mathbf{y}$, if

$$\begin{aligned} x_1^\downarrow &\le y_1^\downarrow \\ x_1^\downarrow + x_2^\downarrow &\le y_1^\downarrow + y_2^\downarrow \\ \cdots &\le \cdots \\ x_1^\downarrow + \cdots + x_k^\downarrow &\le y_1^\downarrow + \cdots + y_k^\downarrow \\ \cdots &\le \cdots \\ x_1 + \cdots + x_n &= y_1 + \cdots + x_n \end{aligned}$$

$$|\Psi\rangle = \sqrt{0.50}\,|1_A 1_B\rangle + \sqrt{0.30}\,|2_A 2_B\rangle + \sqrt{0.20}\,|3_A 3_B\rangle$$
$$|\Phi\rangle = \sqrt{0.60}\,|u_A u_B\rangle + \sqrt{0.25}\,|v_A v_B\rangle + \sqrt{0.15}\,|w_A w_B\rangle$$

$\lambda_\Psi^\downarrow$	$\lambda_\Phi^\downarrow$
0.50	0.60
0.30	0.25
0.20	0.15

$\Longrightarrow$

$\sum^k \lambda_\Psi^\downarrow$		$\sum^k \lambda_\Phi^\downarrow$
0.50	$\leq$	0.60
0.80	$\leq$	0.85
1	$=$	1

$\Longrightarrow$

$$\lambda_\Psi \prec \lambda_\Phi$$

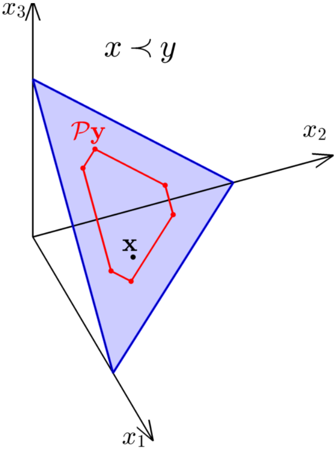
Yes! It is possible to convert $|\Psi\rangle$ to $|\Phi\rangle$ by LOCC.

If $\mathbf{x} \prec \mathbf{y}$ then $\mathbf{x}$ is a convex combination of some permutations of $\mathbf{y}$.

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \prec \mathbf{x} \prec (1, 0, \ldots, 0)$$

More on majorization

- MA Nielsen, G Vidal, Quantum Info & Comp **1**, 76 (2001).
- R Bhatia, *Matrix Analysis*, Springer-Verlag, 1997.



Bipartite pure: Single Copy protocol

Example: $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow[LOCC]{} |\Phi\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$

- Ali does a local *generalized measurement* on A with

$$M_1 = \begin{bmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{bmatrix} \quad M_2 = \begin{bmatrix} \sin\theta & 0 \\ 0 & \cos\theta \end{bmatrix} \qquad M_1^\dagger M_1 + M_2^\dagger M_2 = \mathbb{1}$$

- Two outcomes
  - Prob=1/2,
    - final state: $M_1 \otimes \mathbb{1}\,|\Psi\rangle = (\cos\theta\,|00\rangle + \sin\theta\,|11\rangle)/\sqrt{2} = |\Phi\rangle/\sqrt{2}$.
    - They already got the desired state
    - Do nothing!
  - Prob=1/2,
    - final state: $M_2 \otimes \mathbb{1}\,|\Psi\rangle = (\sin\theta\,|00\rangle + \cos\theta\,|11\rangle)/\sqrt{2}$.
    - Ali calls Berna, tells the result.
    - Both carry out a σ_x rotation on their qubits.
    - $\longrightarrow \sigma_x \otimes \sigma_x\,|\cdots\rangle = (\cos\theta\,|00\rangle + \sin\theta\,|11\rangle)/\sqrt{2} = |\Phi\rangle/\sqrt{2}$

For bipartite pure case: Transformation can be achieved by one-way communication. HK Lo, S Popescu, PRA **63**, 022301 (2001).

Initial state is $|\Psi\rangle$

Final state is $\begin{cases} |\Phi_1\rangle & \text{with probability } p_1 \\ |\Phi_2\rangle & \text{with probability } p_2 \\ \dots & \end{cases}$

Probabilistic - Single Copy

$$|\Psi\rangle \xrightarrow[LOCC]{} \{p_i, |\Phi_i\rangle\} \quad \Longleftrightarrow \quad \lambda_\Psi \prec \sum_i p_i \lambda^{\downarrow}_{\Phi_i}$$

D Jonathan, MB Plenio, PRL **83**, 1455 (1999).

Bipartite pure: Probabilistic Example

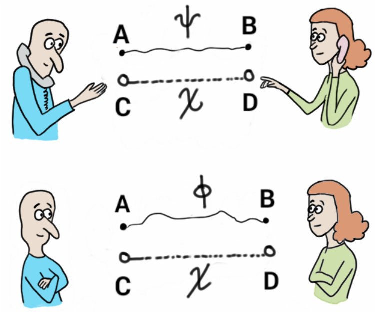
$$|\Psi\rangle = \sqrt{0.50}\,|1_A 1_B\rangle + \sqrt{0.30}\,|2_A 2_B\rangle + \sqrt{0.20}\,|3_A 3_B\rangle$$

$$\xrightarrow[LOCC]{} \begin{cases} \frac{1}{\sqrt{3}}\Big(|1_A 1_B\rangle + |2_A 2_B\rangle + |3_A 3_B\rangle\Big) & \text{with probability } p_3 \\ \frac{1}{\sqrt{2}}\Big(|1_A 1_B\rangle + |2_A 2_B\rangle\Big) & \text{with probability } p_2 \\ |1_A 1_B\rangle & \text{with probability } p_1 \end{cases}$$

$$\lambda_\Psi = (0.5, 0.3, 0.2) \prec p_3\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + p_2\left(\frac{1}{2}, \frac{1}{2}, 0\right) + p_1\Big(1, 0, 0\Big)$$

$$\prec \left(\frac{p_3}{3} + \frac{p_2}{2} + p_1, \frac{p_3}{3} + \frac{p_2}{2}, \frac{p_3}{3}\right)$$

$$\longrightarrow \quad \begin{aligned} p_3 &= 0.6 \\ p_2 &= 0.2 \\ p_1 &= 0.2 \end{aligned}$$



Some transformations can be enabled by the presence of another entangled pair.
D Jonathan, MB Plenio, PRL **83**, 3566 (1999).
Catalysis: The entanglement of the pair CD is used, but not consumed.

$$|\Psi\rangle = \sqrt{0.4}\,|11\rangle + \sqrt{0.4}\,|22\rangle + \sqrt{0.1}\,|33\rangle + \sqrt{0.1}\,|44\rangle$$
$$|\Phi\rangle = \sqrt{0.5}\,|11\rangle + \sqrt{0.25}\,|22\rangle + \sqrt{0.25}\,|33\rangle$$

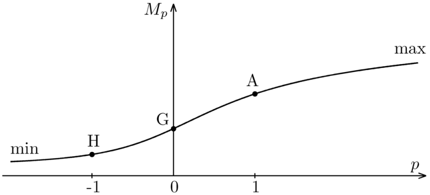
$$|\Psi\rangle \underset{LOCC}{\not\longrightarrow} |\Phi\rangle \qquad \text{where} \quad |\chi\rangle_{CD} = \sqrt{0.6}\,|55\rangle + \sqrt{0.4}\,|66\rangle$$

$$|\Psi \otimes \chi\rangle \underset{LOCC}{\longrightarrow} |\Phi \otimes \chi\rangle$$

Question: Given $|\Psi\rangle$ and $|\Phi\rangle$, is there a catalyst that will enable us to transform $|\Psi\rangle$ and $|\Phi\rangle$?

Power Means:

$$M_p(x) = \left(\frac{x_1^p + \cdots + x_n^p}{n} \right)^{1/p}$$



Catalysis

$$|\Psi\rangle \xrightarrow[cat\&LOCC]{} \Phi \quad \Longleftrightarrow \quad \begin{cases} M_p(\Psi) > M_p(\Phi) & p < 1, \\ M_p(\Psi) < M_p(\Phi) & p > 1, \\ E(\Psi) > E(\Phi) & \end{cases}$$

S Turgut, J Phys A: Math Theor **40**, 12185 (2007).
M Klimesh, arXiv:0709.3680 (2007).

Open: Probabilistic transformations with catalysis

Power Means:

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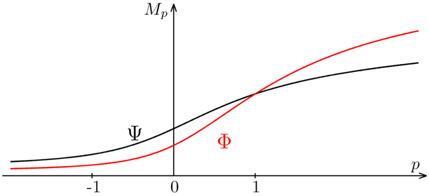

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S Turgut, J Phys A: Math Theor **40**, 12185 (2007).
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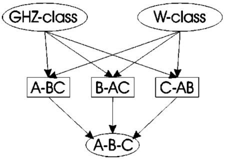
Open: Probabilistic transformations with catalysis

- **Stochastic Reducibility:** $|\Psi\rangle \xrightarrow[SLOCC]{} |\Phi\rangle$ if $|\Phi\rangle$ can be obtained with non-zero probability.
- **Stochastic Equivalence:** $|\Psi\rangle \xleftrightarrow[SLOCC]{} |\Phi\rangle$

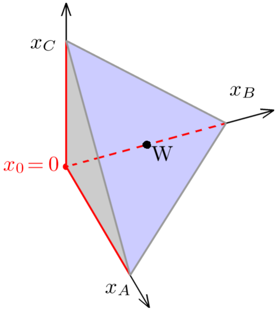
$$\iff \quad |\Psi\rangle_{AB\cdots D} = (X \otimes Y \otimes \cdots \otimes Z)\,|\Phi\rangle_{AB\cdots D} \quad (X, Y, \ldots, Z \text{ invertible}).$$

- **Bipartite States:** Number of non-zero Schmidt coeff. (SLOCC class).
- **Bipartite States:** W Dür *et al.*, PRA **62**, 062314 (2000).

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$
$$|W\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$



Multipartite: W-type states



- p parties holding p qubits ($p \geq 3$).

A W-type state of 3 qubits.

$$|\Psi\rangle_{ABC} = \sqrt{x_0}\,|000\rangle + \sqrt{x_A}\,|100\rangle + \sqrt{x_B}\,|010\rangle + \sqrt{x_C}\,|001\rangle$$

$$x_0 + x_A + x_B + x_C = 1$$

W-type, deterministic

$$\left.\begin{array}{c} |\Psi\rangle_{ABC} \xrightarrow[LOCC]{} |\Psi'\rangle_{ABC} \\ (x_A, x_B, x_C) \xrightarrow[LOCC]{} (x'_A, x'_B, x'_C) \end{array}\right\} \iff \left\{\begin{array}{l} x_A \geq x'_A \\ x_B \geq x'_B \\ x_C \geq x'_C \end{array}\right.$$

S Kıntaş, S Turgut, J Math Phys **51**, 092202 (2010)

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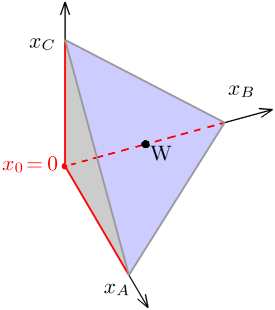
- If x_A must change, then Ali must do the local measurements for this transformation.
- If all $x_A, x_B, \ldots$ change, then all parties must do local measurements.
- Deterministic: Protocol can be completed with a single tour.
- All maximally entangled states are on the $x_0 = 0$ facet.

  $$|\psi\rangle = \sqrt{x_A}\,|100\rangle + \sqrt{x_B}\,|010\rangle + \sqrt{x_C}\,|001\rangle$$

  None of these maximally entangled states can be converted into the others deterministically.
- Probabilistic transformations:
  - The deficit parameter x_0 increases on the average.
  - The other components x_k decrease on the average.



Multipartite state with Schmidt rank 2

$$|\psi\rangle = \frac{1}{N}\left(|00\cdots 0\rangle + z\,|\beta_A\beta_B\cdots\beta_D\rangle\right)$$

- $c_i = \langle 0\,|\beta_i\rangle$ the ith cosine. ($|\beta_i\rangle = c_i\,|0\rangle + \sqrt{1-c_i^2}\,|1\rangle$, $i = A, B, \ldots, D$.)
  - $c_i = 1$ (angle=0°), ith qubit is not entangled with the rest
  - $c_i = 0$ (angle=90°), ith qubit has the "largest" entanglement with the rest
- The state can be parameterized by $(z; c_A, c_B, \ldots, c_D)$
- $(z; c_A, c_B, \ldots, c_D) \xleftrightarrow[LU]{} (1/z; c_A, c_B, \ldots, c_D)$
- If there is a vanishing cosine: $(z; c_A, \ldots, c_D) \xleftrightarrow[LU]{} (|z|\,; c_A, \ldots, c_D)$

Multipartite state with Schmidt rank 2

$$|\psi\rangle = \frac{1}{N}\left(|00\cdots 0\rangle + z\,|\beta_A\beta_B\cdots\beta_D\rangle\right)$$

Deterministic Rank-2 transformation: a few rules

- $\xi = c_A c_B \cdots c_D \dfrac{z+z^*}{1+|z|^2}$ is preserved. (FM Spedalieri, arXiv:quant-ph/0110179)
- Cosines can never decrease ($c_i \longrightarrow c_i' \geq c_i$)

$$\left.\begin{array}{c} |\Psi\rangle \xrightarrow[LOCC]{} |\Psi'\rangle \\ (z; c_A, c_B, \ldots, c_D) \xrightarrow[LOCC]{} (z'; c_A', c_B', \ldots, c_D') \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \xi = \xi' \\ 1 \leq |z| \leq |z'| \\ c_A \leq c_A' \\ \vdots \quad \vdots \\ c_D \leq c_D' \end{array}\right.$$

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Multipartite: Rank 2

Deterministic Rank-2 transformations

If both initial and final states have a vanishing cosine: $\xi = 0$.

$$\left.\begin{array}{r} |\Psi\rangle \xrightarrow[LOCC]{} |\Psi'\rangle \\ \\ (z; c_A, c_B, \ldots, c_D) \xrightarrow[LOCC]{} (z'; c'_A, c'_B, \ldots, c'_D) \end{array}\right\} \iff \left\{\begin{array}{l} |z| \leq |z'| \\ c_A \leq c'_A \\ \vdots \quad \vdots \\ c_D \leq c'_D \end{array}\right.$$

Example: $c_A = c'_A = 0$.

$$\frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle\right) \xrightarrow[LOCC]{} \frac{1}{\sqrt{5}} \left(|000\rangle + 2\,|1\beta_B\beta_C\rangle\right)$$

Deterministic Rank-2 transformations

If initial state has vanishing cosine, final state hasn't.

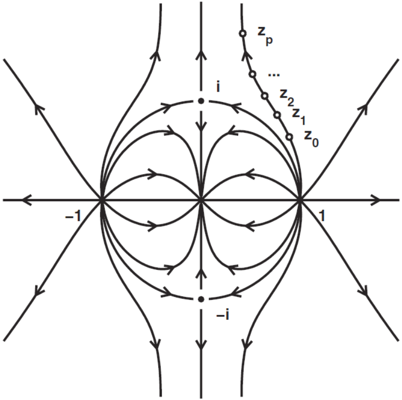
$$\left.\begin{array}{r} |\Psi\rangle \xrightarrow[LOCC]{} |\Psi'\rangle \\ (z; c_A, \ldots, c_D) \xrightarrow[LOCC]{} (z'; c'_A, \ldots, c'_D) \end{array}\right\} \iff \left\{\begin{array}{c} |z| = 1 \\ z' \text{ purely imaginary} \\ c_A \leq c'_A \\ \vdots \quad \vdots \\ c_D \leq c'_D \end{array}\right.$$

Example:

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad \xrightarrow[LOCC]{} \quad \frac{1}{\sqrt{5}}(|000\rangle + 2i\,|\beta_A\beta_B\beta_C\rangle)$$

If both the initial and the final state has no vanishing cosines:

The rule that $\xi = c_A c_B \cdots c_D \dfrac{z+z^*}{1+|z|^2}$ is preserved is enough to determine the change in z



In conclusion

- Entanglement transformation problem has rich mathematics in it.
- Bipartite pure state case has been resolved almost completely.
- Open problems exist in multipartite/mixed cases.
- Complete characterization of probabilistic transformations in selected multipartite cases (W-type, rank 2) is missing.
- Characterization of asymptotic multipartite transformations is needed.

Thanks for listening