Classical Method of Images and Quantum Entangled Coherent States

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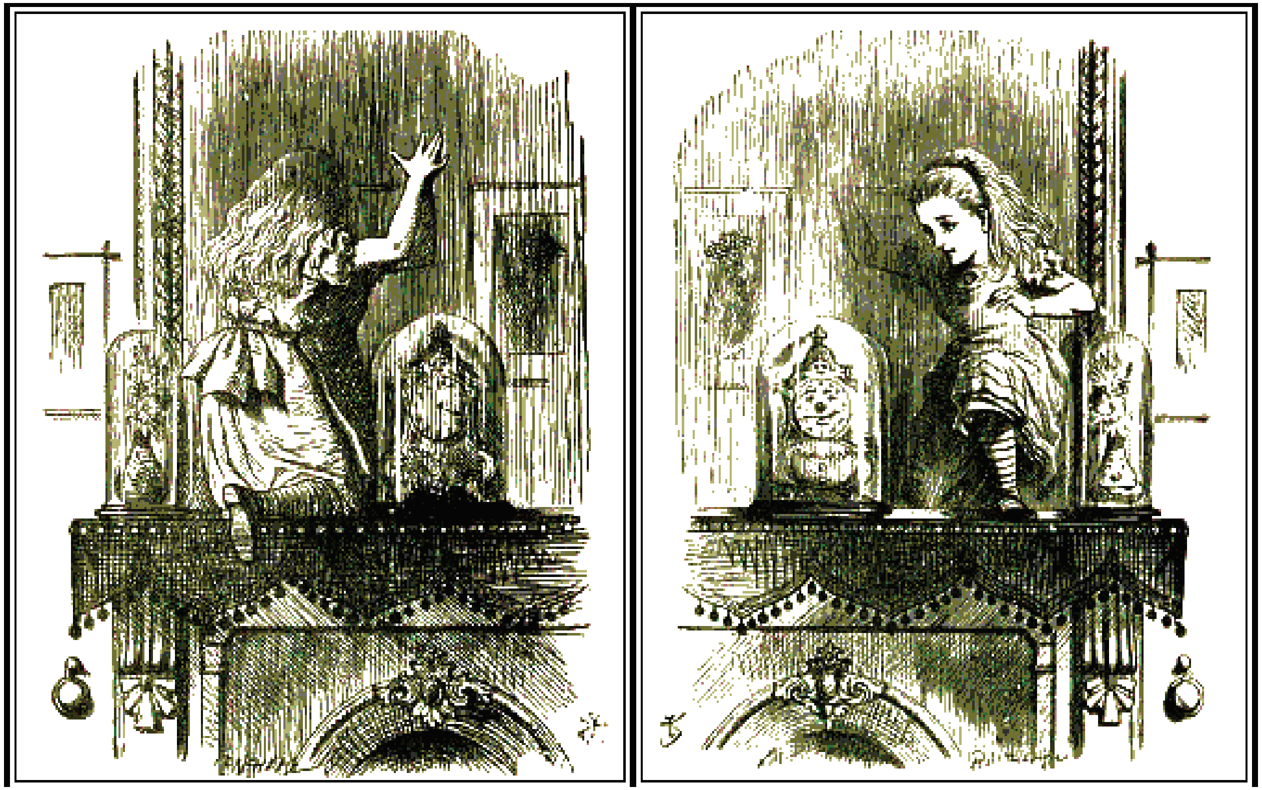
Entangled with Images

Boundary Conditions as origin of images and entanglement of images

Through The Looking Glass



Alice anti-unitary transformation (Wigner)

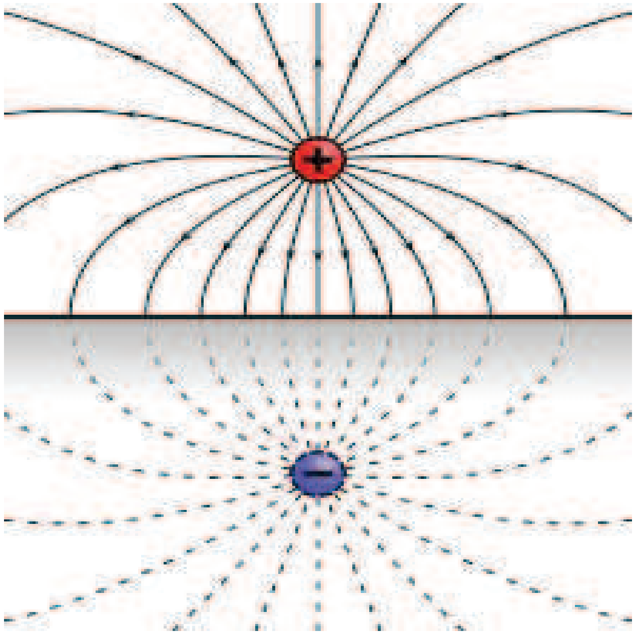
$$K : |\Psi\rangle \rightarrow |\bar{\Psi}\rangle$$

Non Schrödinger Cat Image

not dead and not alive

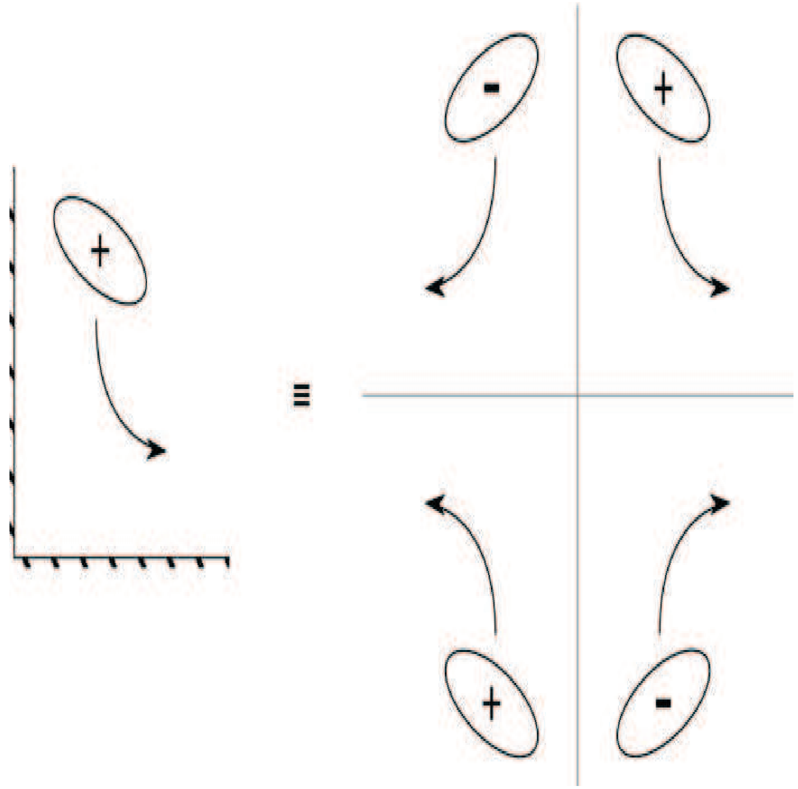
$$Ke^{i\pi} : |\Psi\rangle \rightarrow |-\bar{\Psi}\rangle$$

Electrostatics Images



Boundary Conditions: Potential $U(x, y, z = 0) = 0$

Vortex Images



Point vortex: Boundary Conditions: $Im\, f|_C = 0$

$$f(z) = \Gamma \ln(z - z_0) \;\rightarrow\; \bar{V} = \frac{\Gamma}{z - z_0}$$

Hydrodynamic flow

Bounded domain

For given C-boundary curve find analytic function (complex potential) $F(z)$ with boundary condition

$$\Im F|_C = \psi|_C = 0$$

ψ is the stream function

Normal velocity: $v_n|_C = 0$

Milne-Thomson circle theorem

complex potential

$$F(z) = f(z) + \bar{f}\left(\frac{r^2}{z}\right)$$

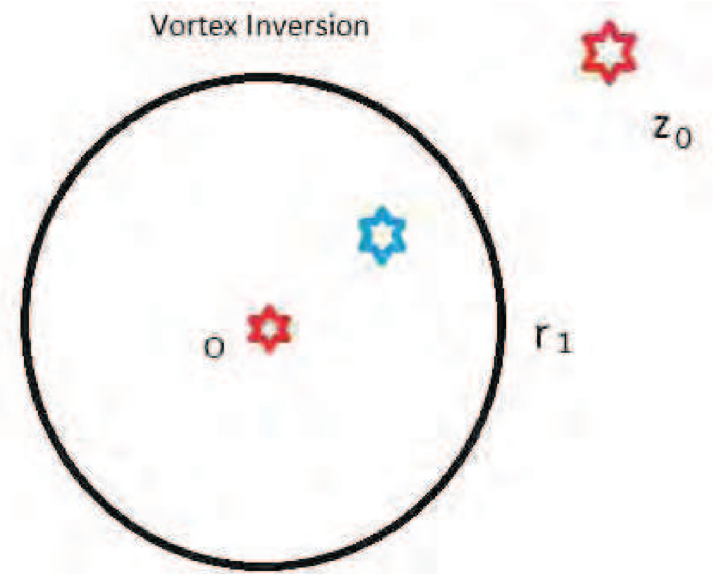
Circle theorem for point vortex

For vortex $+\kappa$ $(\Gamma = -2\pi\kappa)$ at z_0

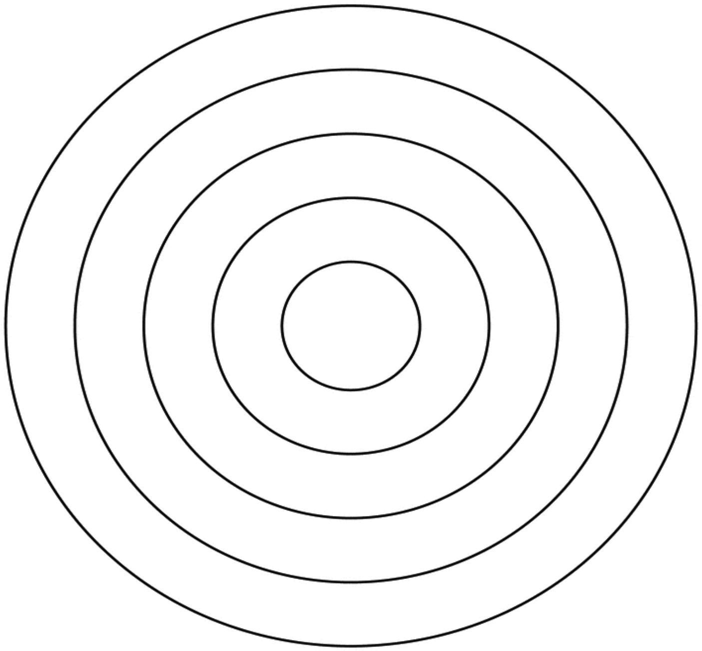
$$\bar{V}(z) = \frac{i\kappa}{z - z_0} - \frac{i\kappa}{z - \frac{r_1^2}{\bar{z}_0}} + \frac{i\kappa}{z}$$

second term - vortex $-\kappa$ at inverse point $r_1^2/\bar{z}_0$ to circle
last term - vortex $+\kappa$ at origin
- vortex images

Vortex Image in the Circle

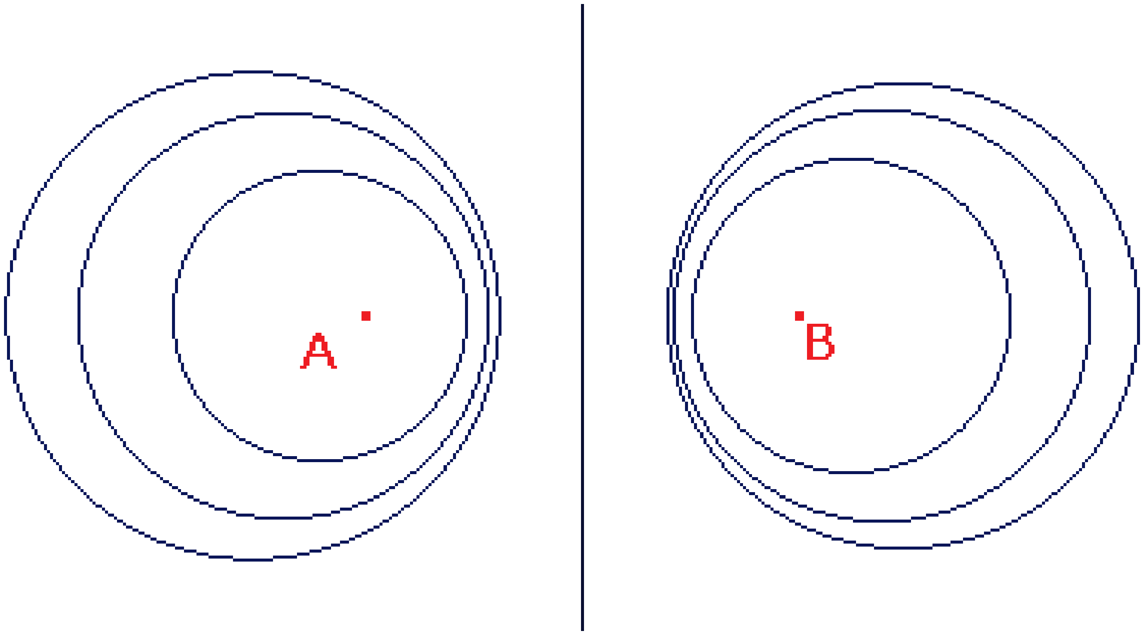


Concentric Circles



Concentric circles: $z = 0$ and $z = \infty$ - common symmetric points

Apollonius Circles



Conformal mapping of concentric circles

$$w = \frac{az + b}{cz + d}$$

A, B - common symmetric points

Two circles theorem

Pashaev 2012: annular domain: $r_1 < |z| < r_2$ between two concentric circles $C_1 : |z| = r_1$, $C_2 : |z| = r_2$

$$F_q(z) = f_q(z) + \bar{f}_q\left(\frac{r^2}{z}\right)$$

where $q = \frac{r_2^2}{r_1^2}$,
$f_q(z) = \sum_{n=-\infty}^{\infty} f(q^n z)$- flow in even annulus
$\bar{f}_q\left(\frac{r^2}{z}\right) = \sum_{n=-\infty}^{\infty} \bar{f}\left(q^n \frac{r^2}{z}\right)$- flow in odd annulus
Proof: $Im\,F(z)|_{C_1} = 0$ and $Im\,F(z)|_{C_2} = 0$

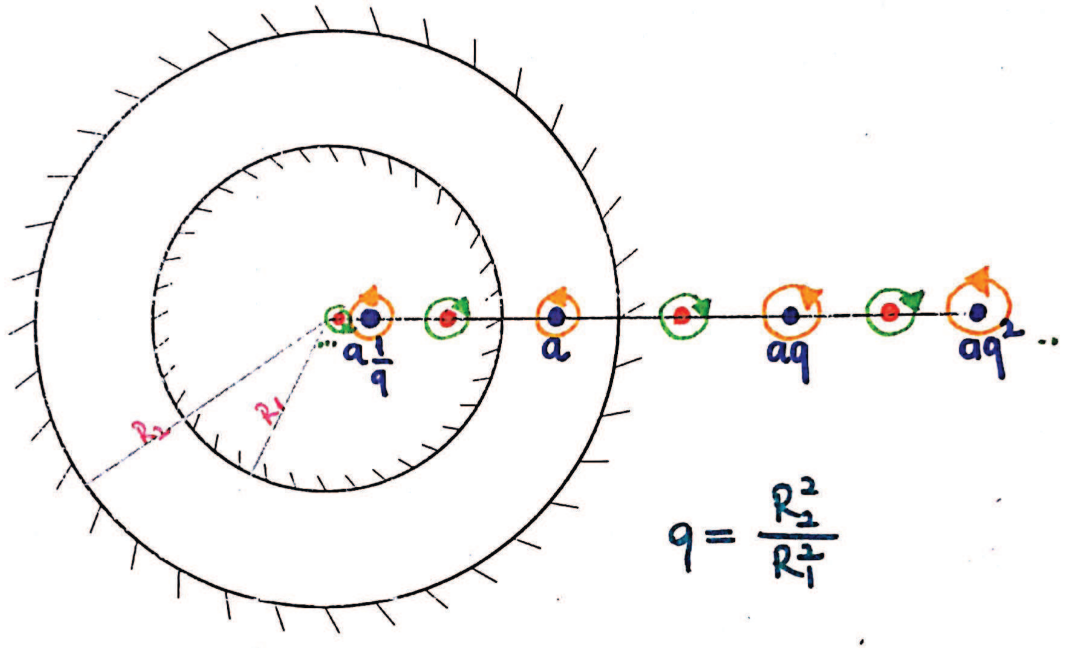
Vortex in annular domain

for vortex at z_0 by Two Circle Theorem

$$F(z) = \frac{\Gamma}{2\pi i} \sum_{n=-\infty}^{\infty} \ln \frac{z - z_0 q^n}{z - \frac{r_1^2}{\bar{z}_0} q^n},$$

$$\bar{V}(z) = \frac{\Gamma}{2\pi i} \sum_{n=-\infty}^{\infty} \left[\frac{1}{z - z_0 q^n} - \frac{1}{z - \frac{r_1^2}{\bar{z}_0} q^n} \right].$$

Vortex images and q-lattice



q - lattice of hydrodynamic vortex images
in annular domain

N -vortex Dynamics

N - point vortices with circulations Γ_1,...,Γ_N, at $z_1, ..., z_N$:

$$\dot{\bar{z}}_k = \frac{1}{2\pi i} \sum_{j=1 (j \neq k)}^{N} \frac{\Gamma_j}{z_k - z_j} + \frac{1}{2\pi i} \sum_{j=1}^{N} \sum_{n=\pm 1}^{\pm\infty} \frac{\Gamma_j}{z_k - z_j q^n}$$

$$- \frac{1}{2\pi i} \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} \frac{\Gamma_j}{z_k - \frac{r_1^2}{\bar{z}_k} q^n}$$

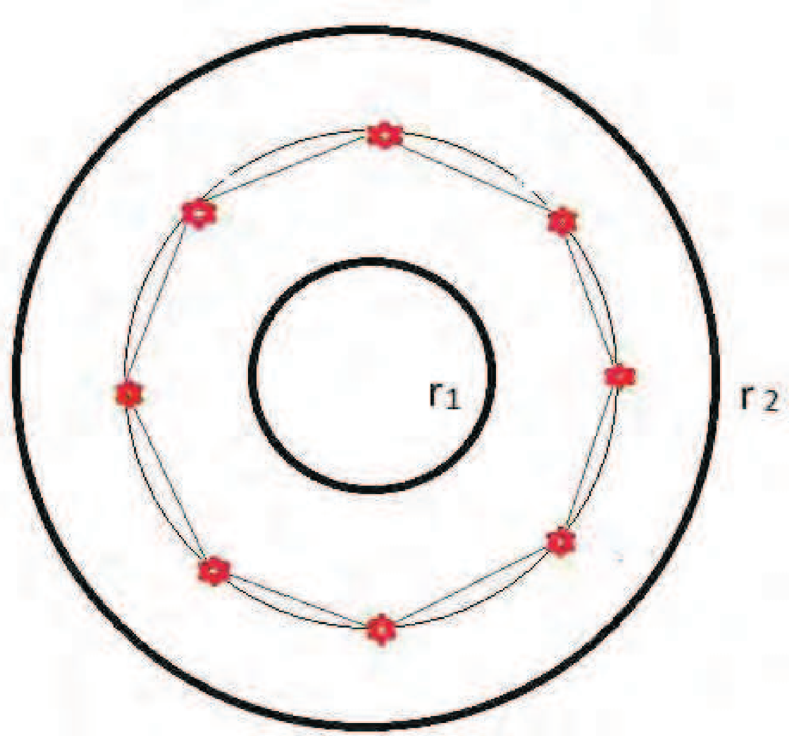
q -Exponential form

Hamiltonian

$$H = -\frac{1}{4\pi} \sum_{i,j=1 (i \neq j)}^{N} \Gamma_i \Gamma_j \ln |z_i - z_j| -$$

$$-\frac{1}{4\pi} \sum_{i,j=1}^{N} \Gamma_i \Gamma_j \ln \left| \frac{e_q\left(\frac{z_i}{(1-q) z_j}\right) e_q\left(\frac{z_j}{(1-q) z_i}\right)}{e_q\left(\frac{z_i \bar{z}_j}{(1-q) r_1^2}\right) e_q\left(\frac{r_2^2}{(1-q) z_i \bar{z}_j}\right)} \right|$$

N Vortex Polygon Solution



$$q = \frac{r_2^2}{r_1^2} > 1$$

N -Polygon Solution

N vortices $\Gamma_k = \Gamma, k = 1, ..., N$, located at the same distance $r_1 < r < r_2$

$$z_k(t) = r e^{i\omega t + i\frac{2\pi}{N}k}$$

rotation frequency

$$\frac{2\pi r^2 (q-1)}{\Gamma}\omega = \frac{N-1}{2}(q-1) +$$

$$\sum_{j=1}^{N}\left[Ln_q\left(1 - \frac{r_2^2}{r^2}e^{i\frac{2\pi}{N}j}\right) - Ln_q\left(1 - \frac{r^2}{r_1^2}e^{-i\frac{2\pi}{N}j}\right)\right]$$

Wedge theorem

For given flow $f(z)$, introduction of boundary wedge with angle $\alpha = 2\pi/N = \pi/n$, $N = 2n$ - positive even number, produces flow

$$F_q(z) = f(z) + f(q^2 z) + f(q^4 z) + ... + f(q^{2(n-1)} z)$$

$$+\bar{f}(z) + \bar{f}(q^2 z) + \bar{f}(q^4 z) + ... + \bar{f}(q^{2(n-1)} z)$$

or shortly

$$F_q(z) = \sum_{k=0}^{n-1} f(q^{2k} z) + \sum_{k=0}^{n-1} \bar{f}(q^{2k} z)$$

$q = e^{i\frac{2\pi}{N}} = e^{i\frac{\pi}{n}}$ primitive root of unity $q^N = 1$
Pashaev (2014)

Kaleidoskop

Kaleidoskop

Vortex kaleidoscope

for

$$f(z) = \frac{i\Gamma}{2\pi} \ln(z - z_0)$$

we have

$$F_q(z) = \frac{i\Gamma}{2\pi} \sum_{k=0}^{n-1} \ln \frac{z - z_0 q^{2k}}{z - \bar{z}_0 q^{2k}} = \frac{i\Gamma}{2\pi} \ln \prod_{k=0}^{n-1} \frac{z - z_0 q^{2k}}{z - \bar{z}_0 q^{2k}}$$

kaleidoscope of $2n$ vortices: positive strength at $z_0, z_0 q^2, z_0 q^4 \ldots, z_0 q^{2(n-1)}$, negative strength at $\bar{z}_0, \bar{z}_0 q^2, \bar{z}_0 q^4 \ldots, \bar{z}_0 q^{2(n-1)}$.

Vortex kaleidoscope

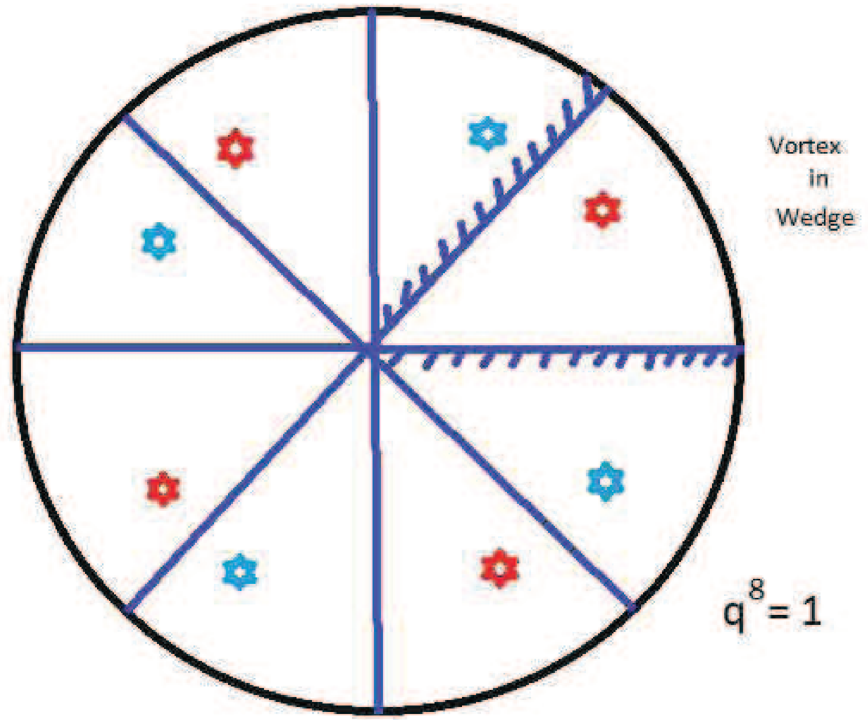
due to identity (E. Kummer):

$$(z - z_0)(z - z_0 q^2)(z - z_0 q^4)\ldots(z - z_0 q^{2(n-1)}) = z^n - z_0^n$$

valid for $q^2 = e^{i\frac{2\pi}{n}}$ as the primitive n-th root of unity, compact expression for the vortex flow in the wedge (the Kummer kaleidoscope of vortices)

$$F_q(z) = \frac{i\Gamma}{2\pi} \ln \frac{z^n - z_0^n}{z^n - \bar{z}_0^n}$$

Vortex in Wedge



$q = e^{i\frac{\pi}{4}}$

$$F_q(z) = \frac{i\Gamma}{2\pi} \ln \frac{z^4 - z_0^4}{z^4 - \bar{z}_0^4}$$

Circular wedge theorem

angle $\alpha = 2\pi/N = \pi/n$, bounded by lines Γ_1: $z = x$ and Γ_2: $z = xe^{i\frac{\pi}{n}}$ and circular boundary $C_1 : z = re^{it}$, $0 < t < \alpha$. Then the flow is

$$F_q(z) = \sum_{k=0}^{n-1}[f(q^{2k}z)+\bar{f}(q^{2k}z)]+\sum_{k=0}^{n-1}[\bar{f}(\frac{r^2}{q^{2k}z})+f(\frac{r^2}{q^{2k}z})]$$

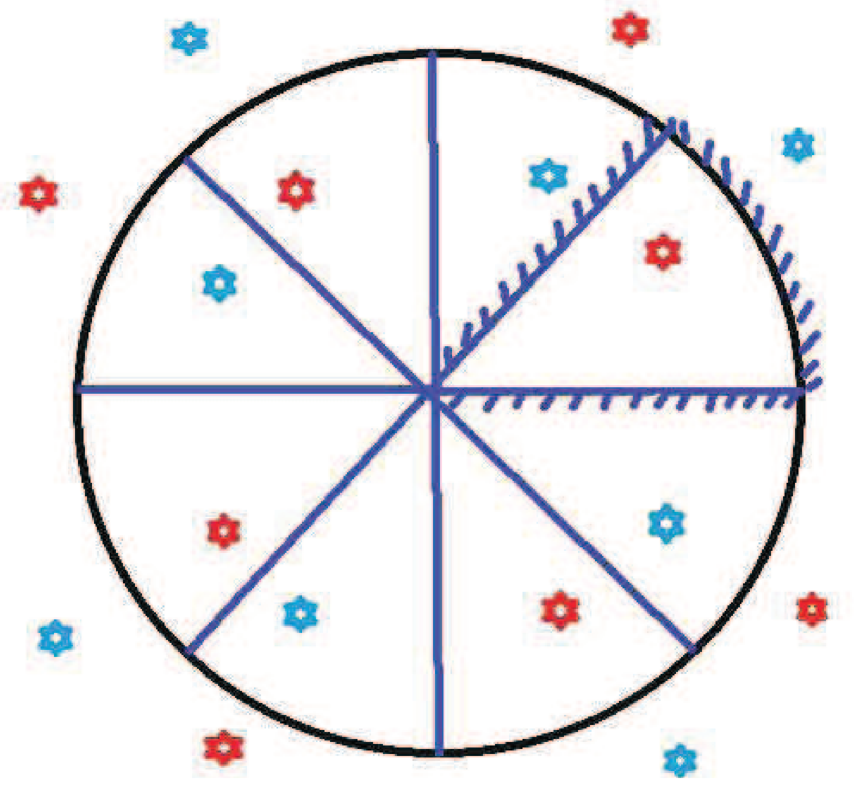
Circular vortex kaleidoscope

single vortex in circular wedge

$$F_q(z) = \frac{i\Gamma}{2\pi} \ln \frac{(z^n - z_0^n)(z^n - \frac{r^{2n}}{z_0^n})}{(z^n - \bar{z}_0^n)(z^n - \frac{r^{2n}}{\bar{z}_0^n})}$$

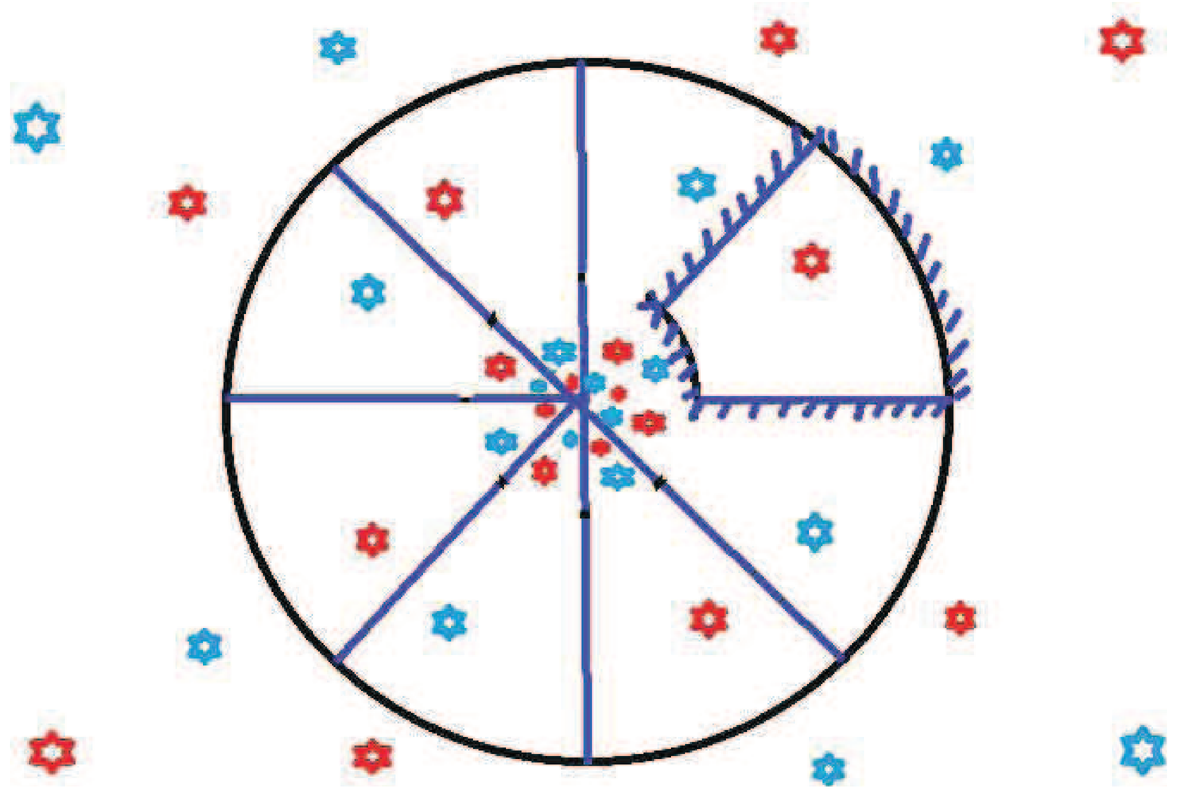
- doubling of images by reflection in circle r.

Vortex in Circular Wedge



$$F_q(z) = \frac{i\Gamma}{2\pi} \ln \frac{(z^4 - z_0^4)(z^4 - \frac{r^8}{z_0^4})}{(z^4 - \bar{z}_0^4)(z^4 - \frac{r^8}{\bar{z}_0^4})}$$

Double Circular Wedge



$$F(z) = \frac{i\Gamma}{2\pi} \sum_{m=-\infty}^{\infty} \ln \frac{(z^4 - z_0^4 Q^{4m})(z^4 - \frac{r_2^8}{z_0^4} Q^{4m})}{(z^4 - \bar{z}_0^4 Q^{4m})(z^4 - \frac{r_2^8}{\bar{z}_0^4} Q^{4m})}$$

Quantum Vortex in Annular Domain

Point vortex problem in annular domain as a nonlinear oscillator or as f-oscillator

$$\dot{z}_0 = -i\omega z_0$$

$$E_n = \frac{\Gamma^2}{4\pi} \ln \left| e_q\left(\frac{(n)}{(1-q)r_1^2}\right) e_q\left(\frac{r_2^2}{(1-q)(n)}\right) \right|$$

$$+\frac{\Gamma^2}{4\pi} \ln \left| e_q\left(\frac{(n+1)}{(1-q)r_1^2}\right) e_q\left(\frac{r_2^2}{(1-q)(n+1)}\right) \right|$$

Images and Quantum States

Method of images $\rightarrow$ construct set of multiple qubit coherent states.

Qubit $\leftrightarrow$ arbitrary point in extended complex plane

Reflected (inverted) qubit $\leftrightarrow$ reflected (inverted) image point

Coherent States

Coherent States and Complex Plane
Coherent States - introduced by Schrodinger (harmonic oscillator), in quantum optics - Glauber (minimization of Heisenberg uncertainty relations - closed to classical states) $\alpha \in \mathcal{C}$ complex number

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Cat states: $|\alpha\rangle \pm |-\alpha\rangle$ - reflection at origin
Entangled coherent states :

$$|\Psi\rangle = |\alpha\rangle|\alpha\rangle \pm |-\alpha\rangle|-\alpha\rangle$$

Spin Coherent States

Spin Coherent States or $SU(2)$ coherent states (quantum optics, etc)

Spin Coherent States (generalized CS) - widely used in quantum information theory $\rightarrow$ studying the entanglement of spin coherent states as a measure of their classicality

Mobius transformation on qubit

Linear transformation

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

for homogeneous coordinates $\psi = \psi_1/\psi_2$, $w = w_1/w_2$ implies Mobius Transformation

$$w = \frac{a\psi + b}{c\psi + d}, \quad ad - bc \neq 0$$

Qubit and coherent state

$$|\psi> = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Rightarrow$$

$$|\psi> = \frac{1}{\sqrt{1+|\psi|^2}} \begin{pmatrix} 1 \\ \psi \end{pmatrix}$$

spin $\frac{1}{2}$ generalized coherent state. Stereographic projection of Bloch sphere to complex plane $\psi \in C$: $\psi = \tan\frac{\theta}{2} e^{i\varphi}$

$$|\theta, \varphi> = \cos\frac{\theta}{2}|0> + \sin\frac{\theta}{2} e^{i\varphi}|1>$$

Symmetric points

ψ and ψ^* - symmetrical with respect to circle C through ψ_1, ψ_2, ψ_3 iff

$$(\psi^*, \psi_1, \psi_2, \psi_3) = \overline{(\psi, \psi_1, \psi_2, \psi_3)}$$

-cross ratio of four points

1. ψ and $\psi^* = \bar{\psi}$ projections of $M(x, y, z)$ and $M^*(x, -y, z)$ reflection in x-axis
2. ψ and $\psi^* = -\bar{\psi} \rightarrow M^*(-x, y, z)$ in y-axis
3. ψ and $\psi^* = \frac{1}{\bar{\psi}} \rightarrow M^*(x, y, -z)$ inverse point
4. ψ and $\psi^* = -\frac{1}{\bar{\psi}} \rightarrow M^*(-x, -y, -z)$ antipodal point

Symmetric qubits

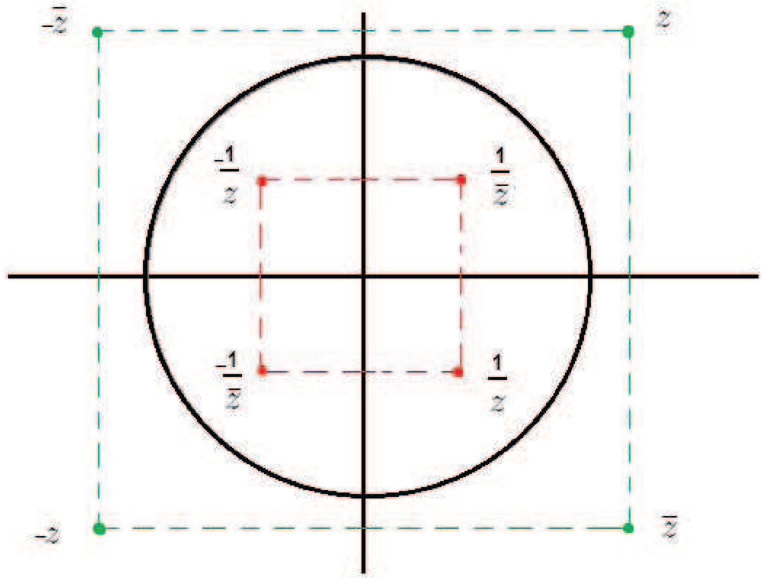
For qubit $|\theta, \varphi >$ symmetric states

1. $|\theta, -\varphi > \rightarrow |\bar{\psi} >= \frac{|0> + \bar{\psi}|1>}{\sqrt{1+|\psi|^2}}$
2. $|\theta, \pi - \varphi > \rightarrow |-\bar{\psi} >= \frac{|0> - \bar{\psi}|1>}{\sqrt{1+|\psi|^2}}$
3. $|\pi - \theta, \varphi > \rightarrow |\frac{1}{\bar{\psi}} >= \frac{\bar{\psi}|0> + |1>}{\sqrt{1+|\psi|^2}}$
4. $|\pi - \theta, \varphi + \pi > \rightarrow |-\frac{1}{\bar{\psi}} >= \frac{-\bar{\psi}|0> + |1>}{\sqrt{1+|\psi|^2}}$

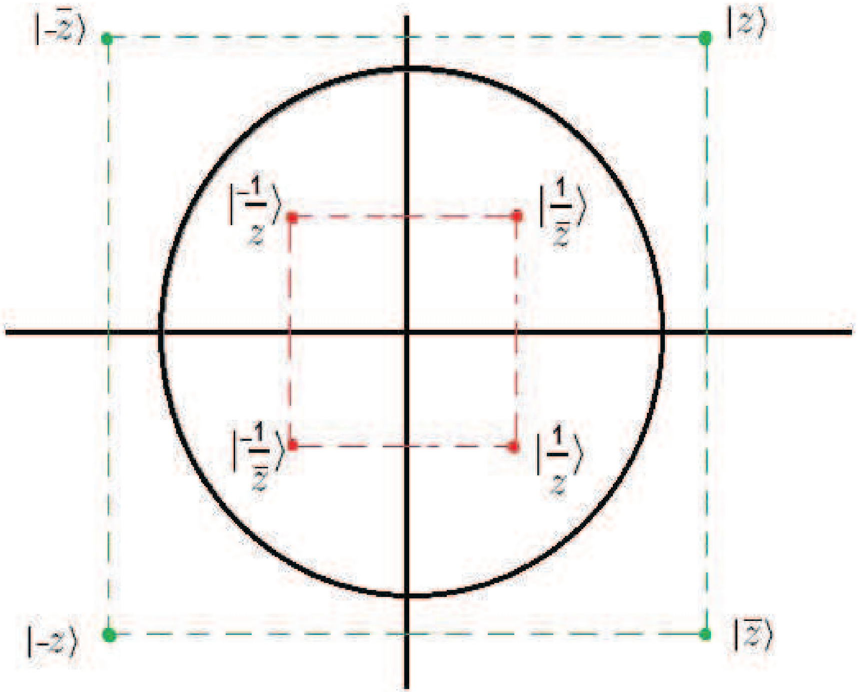
antipodal symmetric coherent states are orthogonal

$$< -\psi^*|\psi >= 0$$

Reflections in Circle and Lines



Symmetric States



Alice Entangled Images

$$|-\bar{A}\rangle \qquad |A\rangle$$

$$|-A\rangle \qquad |\bar{A}\rangle$$

Inversion and Antipodal Images

$|-1/A\rangle \quad |-\bar{A}\rangle \qquad |A\rangle \quad |1/\bar{A}\rangle$

$|-1/\bar{A}\rangle \quad |-A\rangle \qquad |\bar{A}\rangle \quad |1/A\rangle$

Anti-unitary Transformation

Wigner (1932) Schrödinger equation $\rightarrow$ time reflection $t \rightarrow -t$ implies

$$K|\psi\rangle = |\bar{\psi}\rangle$$

complex conjugation of the wave function

$$K^2|\psi\rangle = |\psi\rangle \rightarrow K^2 = I$$

$$K(a|\varphi\rangle + b|\psi\rangle) = \bar{a}K|\phi\rangle + \bar{b}K|\psi\rangle$$

nonlinear operator

Concurence

Wootters (2001) Two qubits concurence

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

$$|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$$

$\Downarrow \; K$

$$|\bar{\psi}\rangle = \bar{c}_{00}|00\rangle + \bar{c}_{01}|01\rangle + \bar{c}_{10}|10\rangle + \bar{c}_{11}|11\rangle$$

$$|\tilde{\psi}\rangle = \sigma_y \otimes \sigma_y |\bar{\psi}\rangle$$

spin-flip operation

$$C_\psi = |\langle\tilde{\psi}|\psi\rangle| = 2|c_{00}c_{11} - c_{01}c_{10}|$$

Spin Flip

spin flip operator

$$K|\mathbf{n}\rangle = |-\mathbf{n}\rangle$$

is not unitary, but anti-unitary
Inversion of the Bloch sphere $\rightarrow$ antipodal state
Can not be realized by nature in exact form

$$|\tilde{\psi}\rangle = \sigma_y|\bar{\psi}\rangle \rightarrow C = |\langle\tilde{\psi}|\psi\rangle|$$

Two qubit coherent states

$|\psi>|\psi>, |\psi>|-\psi^*>, |-\psi^*>|\psi>,$ $|-\psi^*>|-\psi^*>$ $\rightarrow$ maximally entangled set of orthogonal two qubit coherent states $\rightarrow$ Pashaev ,N. Gurkan 2012

$$|P_{\pm}>=\frac{1}{\sqrt{2}}(|\psi>|\psi>\pm|-\psi^*>|-\psi^*>)$$

$$|G_{\pm}>=\frac{1}{\sqrt{2}}(|\psi>|-\psi^*>\pm|-\psi^*>|\psi>)$$

1. reduced density matrix $C=1$
2. concurence by determinant form $C=1$
3. average of spin operators $<S>=0$ $\rightarrow$ entangled states are maximally nonclassical states

Q symbol of Hamiltonian

$$\mathcal{H}(\psi, \psi) =< \psi|H|\psi >$$

average energy in coherent state
For XXZ model

$$H = -J(S_1^+ S_2^- + S_1^- S_2^+) + 2\Delta S_1^z S_2^z$$

$$< P_+|H|P_+ >=$$

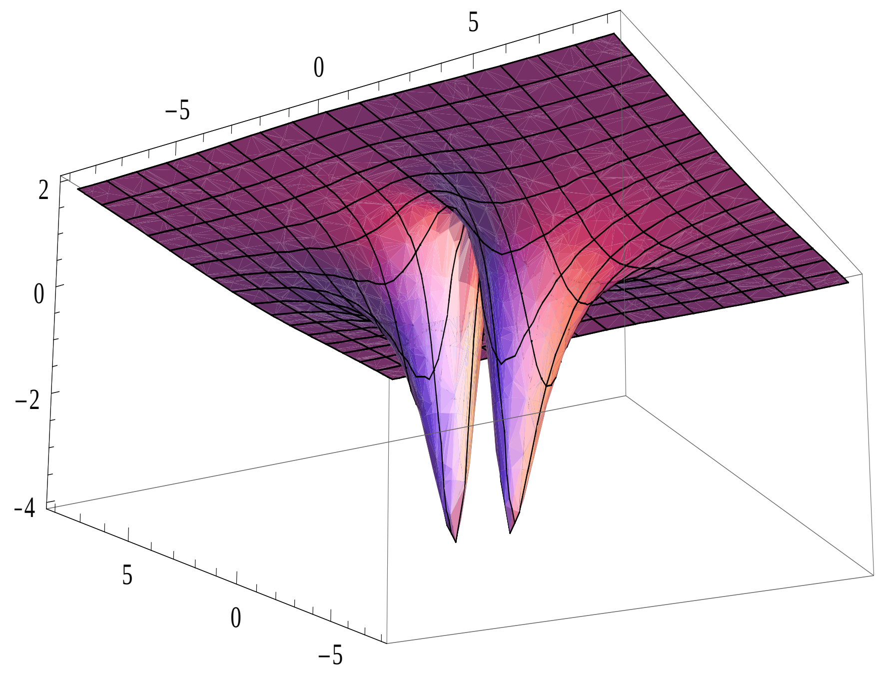
$$-\hbar^2 \frac{8Jy^2 + J_z[1 + 2x^2 - 6y^2 + (x^2 + y^2)^2]}{(1 + x^2 + y^2)^2}$$

where $\psi = \psi_1 + i\psi_2 \equiv x + iy$

2 qubit energy in XXZ model

XXZ average energy surface in maximally entangled $|P_+>$ state

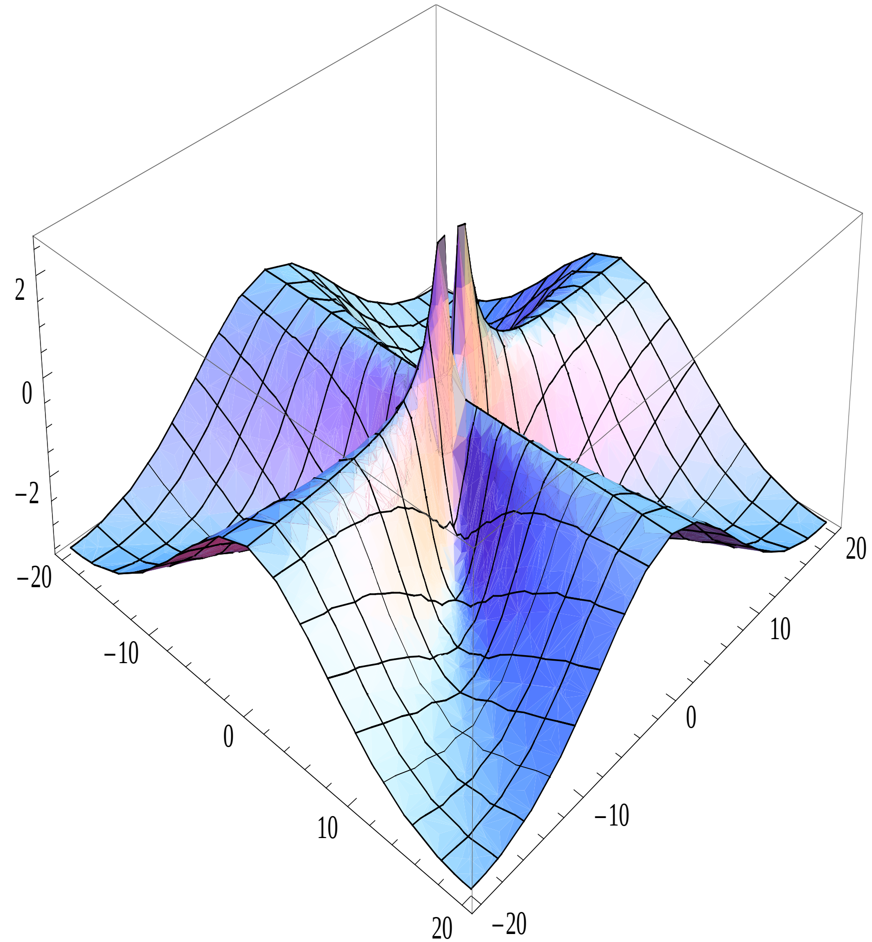
Out[2]=



2 qubit energy in XYZ model

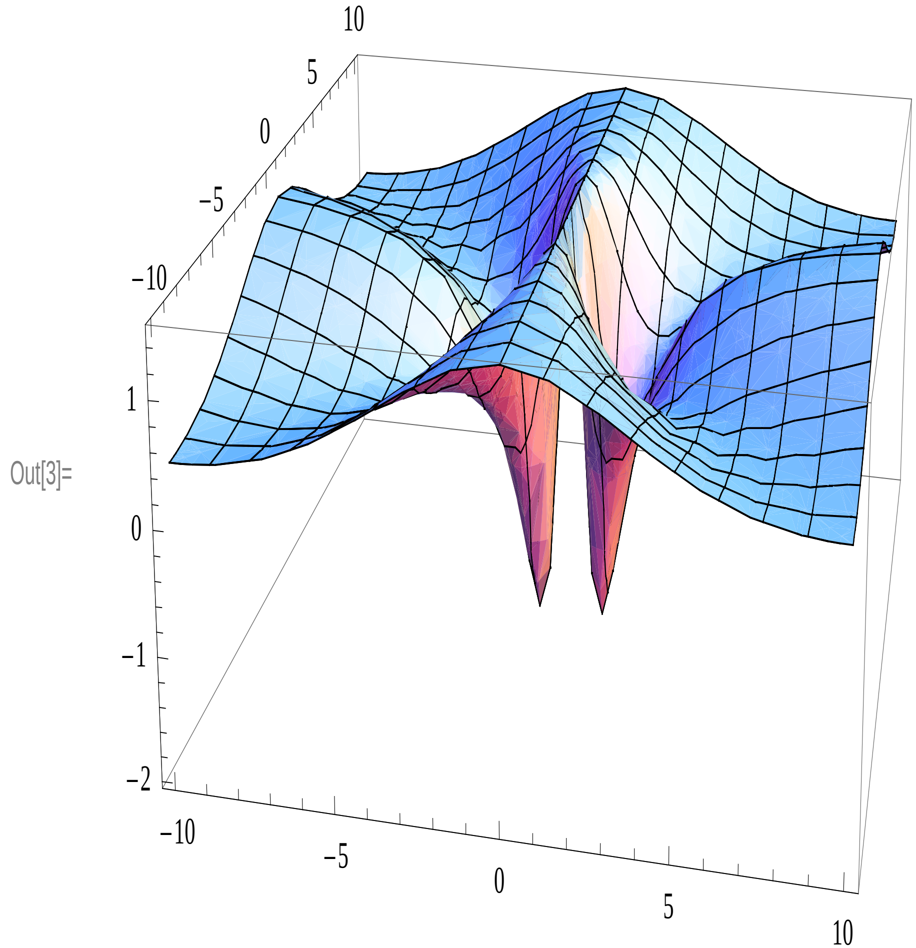
XYZ average energy surface in maximally entangled $|P_+>$ state

Out[8]=



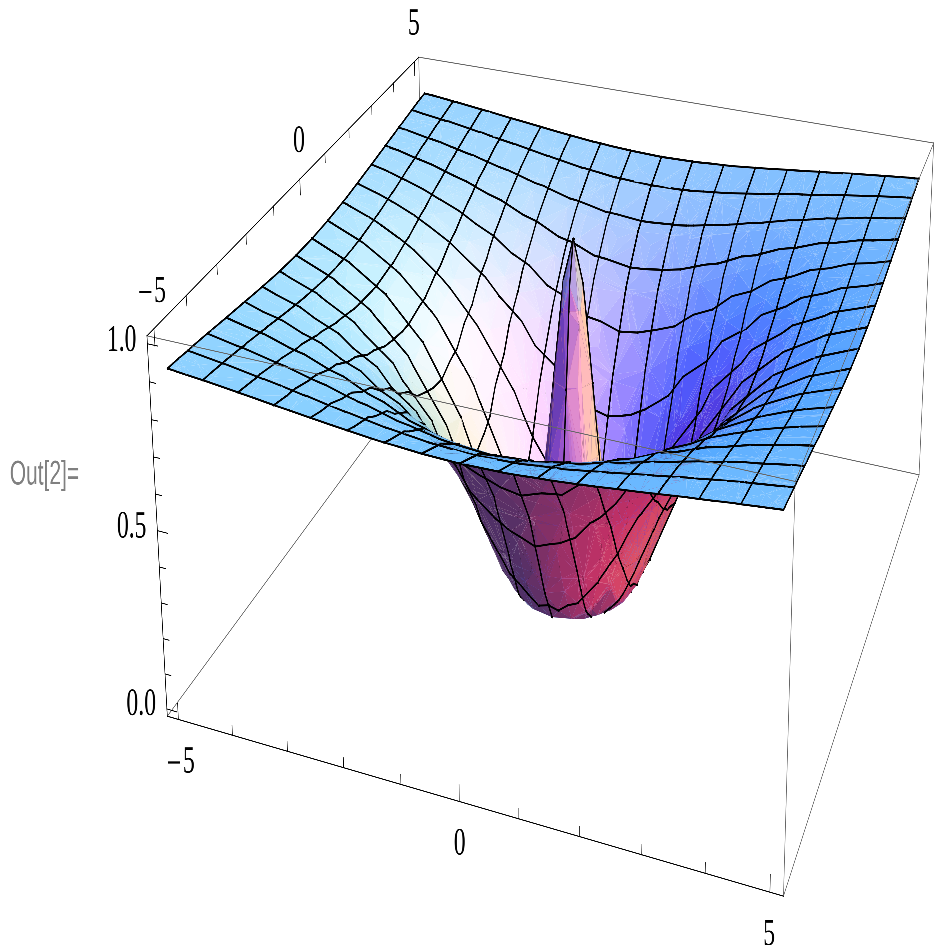
2 qubit energy in XYZ model

XYZ average energy surface in maximally entangled $|P_- >$ state



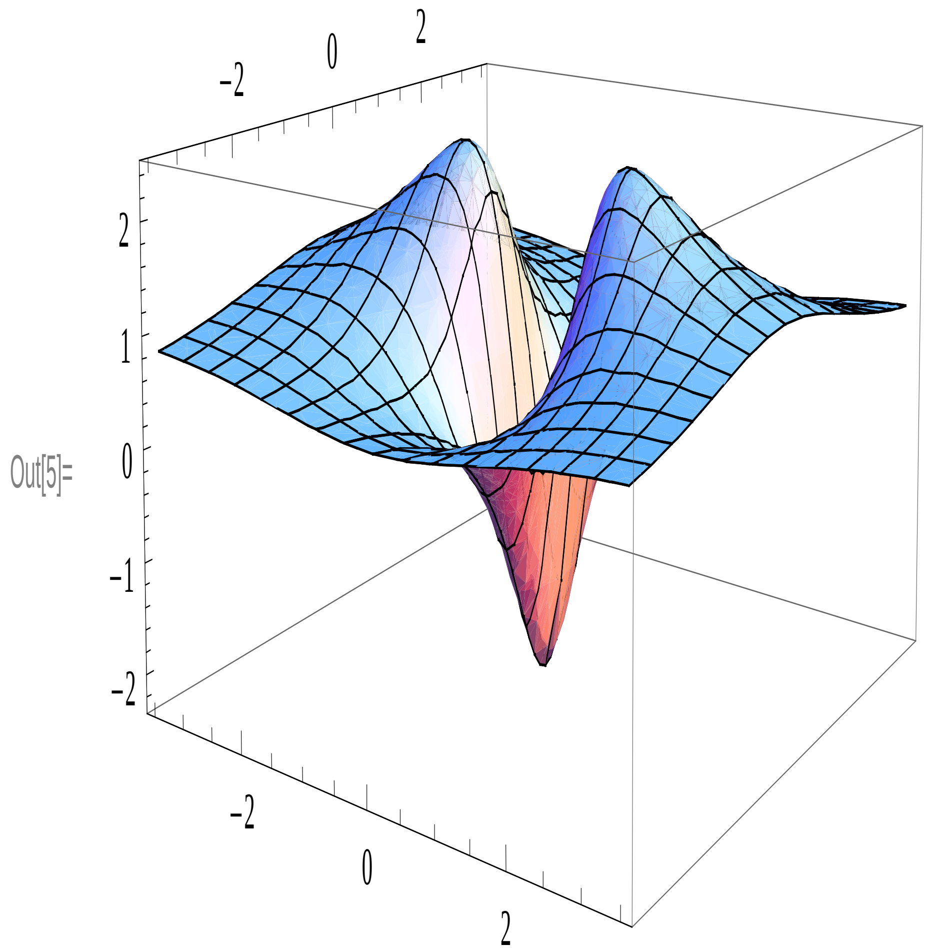
2 qubit energy in XYZ model

XYZ average energy surface in $|G_+>$ state, $J_+ = 1$, $J_{=}0$, $J_z = 0$



2 qubit energy in XYZ model

XYZ average energy surface in $|G_+>$ state,
$J_+ = -1.5$, $J_- = -1.5$, $J_z = 1.5$



3 qubit case in XYZ model

$$|PG_+ >= \frac{|\psi > |\psi > |\psi > + | -\psi^* > | -\psi^* > | -\psi^* >}{\sqrt{2}}$$

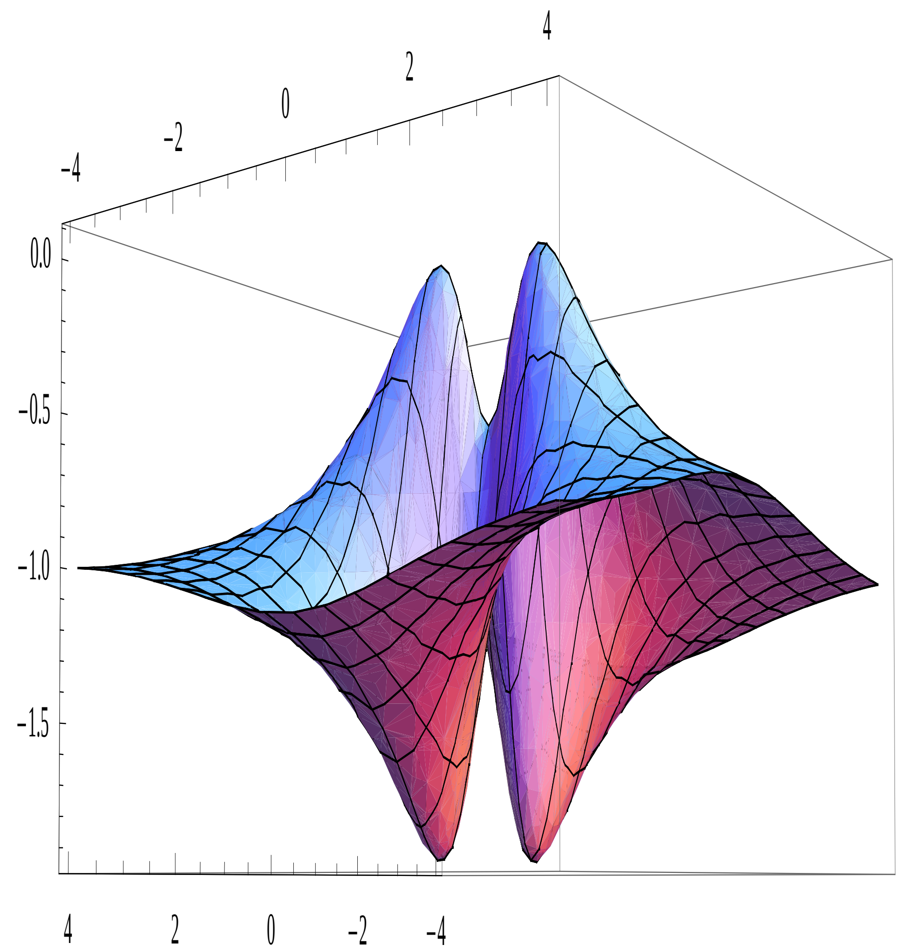
in the limit $\psi \rightarrow 0$, $-\psi^* \rightarrow \infty$ reduced to maximally entangled GHZ state

$$|GHZ >= \frac{1}{\sqrt{2}}(|000 > + |111 >)$$

3 qubit energy in XYZ model

XYZ average energy surface in $|PG_+>$ state, $J_+ = -1$, $J_- = -1$, $J_z = -1$

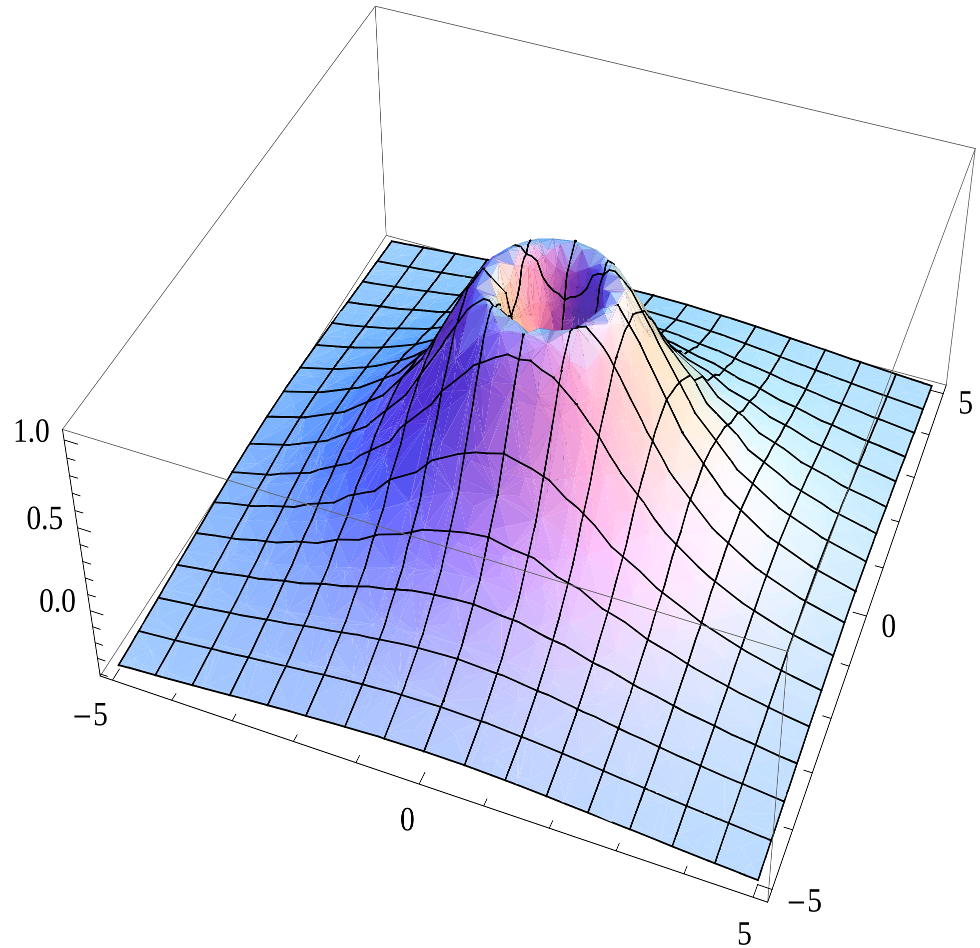
Out[6]=



3 qubit energy in XYZ model

XYZ average energy surface in $|PG_+>$ state,
$J_+ = 1$, $J_- = 0$, $J_z = -0.5$

Out[11]=



Three qubit entangled state

$$|PG_- >= |\psi > |\psi > | - \psi^* > +|\psi > | - \psi^* > |\psi >$$

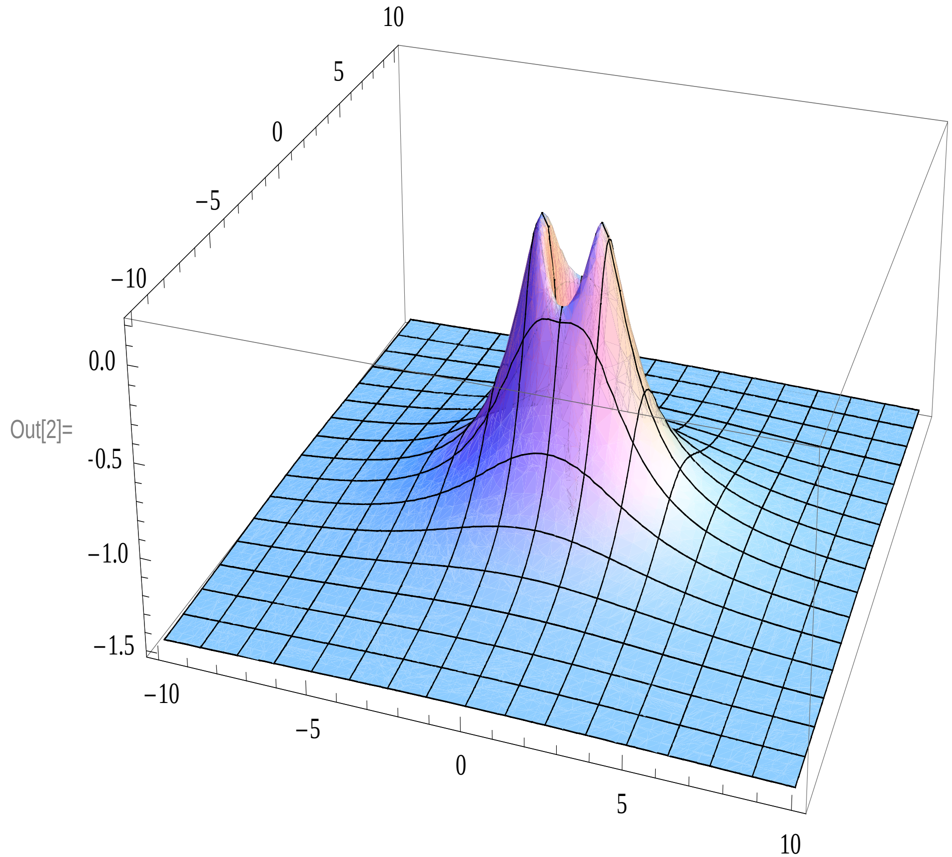
$$+| - \psi^* > |\psi > |\psi >$$

in the limit $\psi \rightarrow 0$, $-\psi^* \rightarrow \infty$ reduced to

$$|W >= \frac{|001 > +|010 > +|100 >}{\sqrt{3}}$$

3 qubit energy in XYZ model

XYZ average energy surface in maximally entangled $|PG_{-}>$ state



Fibonacci Coherent States

$$|\psi>^N = \bigotimes(|0> + \psi|1>)^N$$
$$= |00...0> + \psi(|10...0> + ...|00...1>)$$
$$+\psi^2(|11...0> + ...|00...11>) + ...\psi^N|11...1>)$$

Entangled state $\rightarrow |\mathcal{F}_N(\psi)\rangle = \frac{|\psi>^N - |-\psi^*>^N}{\psi-(-\psi^*)} =$

$$F_1(\alpha,\beta)(|10...0> + ...|00...1>) +$$

$$F_2(\alpha,\beta)(|11...0> + ...|0...11>) +$$

$$...F_N(\alpha,\beta)|11...1>$$

complex Fibonacci polynomials

N qubit Lucas Coherent States

Entangled state

$$|\mathcal{L}_N(\psi)\rangle = |\psi>^N + |-\psi^*>^N = |00...0> +$$

$$L_1(\alpha,\beta)(|10...0> + ...|00...1>) +$$

$$L_2(\alpha,\beta)(|11...0> + ...|0...11>) +$$

$$...L_N(\alpha,\beta)|11...1>$$

complex Lucas polynomials

$$L_n = \varphi^n + (-\varphi)^{-n}$$

Lucas numbers

Complex Fibonacci polynomials

Conjugate points ψ and $-\frac{1}{\bar{\psi}}$ are roots of equation

$$\xi^2 = (\psi - \frac{1}{\bar{\psi}})\xi + \frac{\psi}{\bar{\psi}}$$

$\psi = |\psi|e^{i\varphi}$ and $\xi = \eta e^{i\varphi}$ leads to $\eta^2 = a\eta + 1$,
$a = |\psi| - |\psi|^{-1}$, $\eta^n = \eta F_n(a) + F_{n-1}(a)$
Fibonacci polynomials $F_1(a) = 1$, $F_2(a) = a$

$$F_{n+1}(a) = aF_n(a) + F_{n-1}(a)$$

Binet Formula

Fibonacci polynomials as q-numbers

$$F_n(\eta) = \frac{\eta^n - (-\eta)^{-n}}{\eta - (-\eta)^{-1}} = [n]_{\eta, -\eta^{-1}}$$

inverse-symmetrical q-calculus

For $a = 1 \rightarrow |\psi| = \varphi = \frac{1+\sqrt{5}}{2}$ - Golden Ratio

Fibonacci numbers as q-numbers

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\varphi - (-\varphi)^{-1}} = [n]_F$$

Golden Calculus - Pashaev, S. Nalci, 2012

$$F_{n+1}(\alpha,\beta) = \alpha F_n(\alpha,\beta) + \beta F_{n-1}(\alpha,\beta)$$

$\alpha = \psi - \bar{\psi}^{-1}, \beta = \psi/\bar{\psi}$

$$L_n(\alpha,\beta) = \psi^n + (-\bar{\psi})^{-n}$$

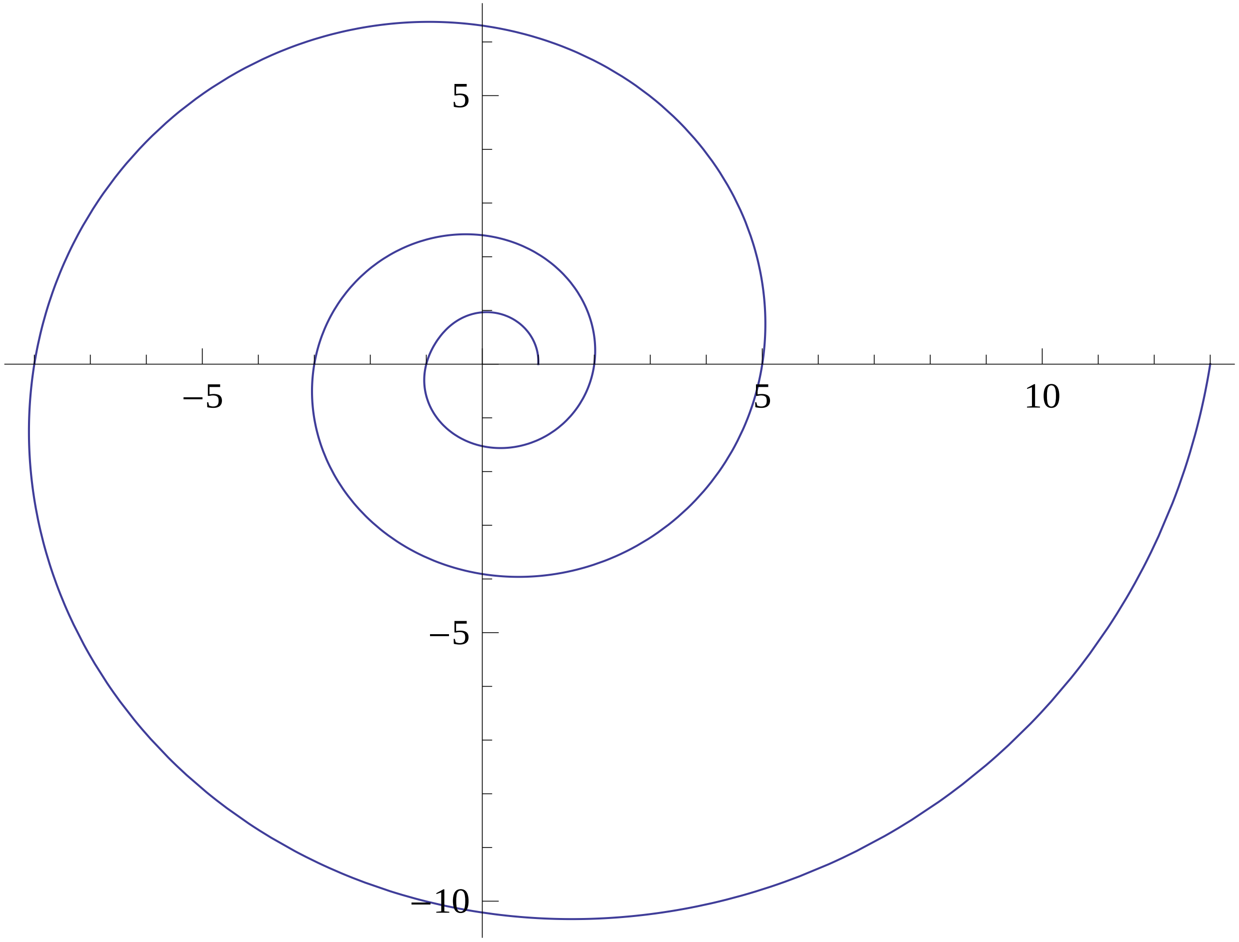
$$F_n(\alpha,\beta) = F_n(a)e^{i\phi(n-1)}$$

$$L_n(\alpha,\beta) = L_n(a)e^{i\phi n}$$

$\phi = arg\,\psi$

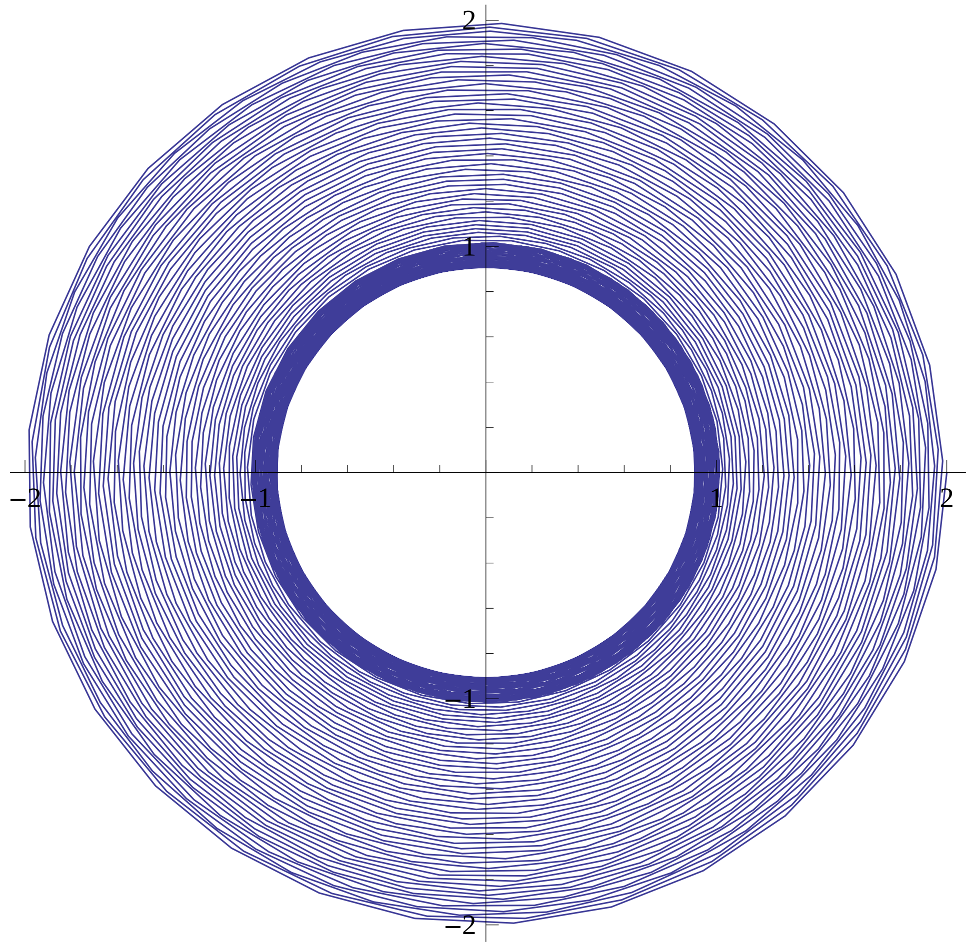
Fibonacci spiral: $\phi = \pi$

Out[118]=



Fibonacci spiral $\phi = 100\pi$: t=3

Out[164]=



Complex Fibonacci spiral: t=4

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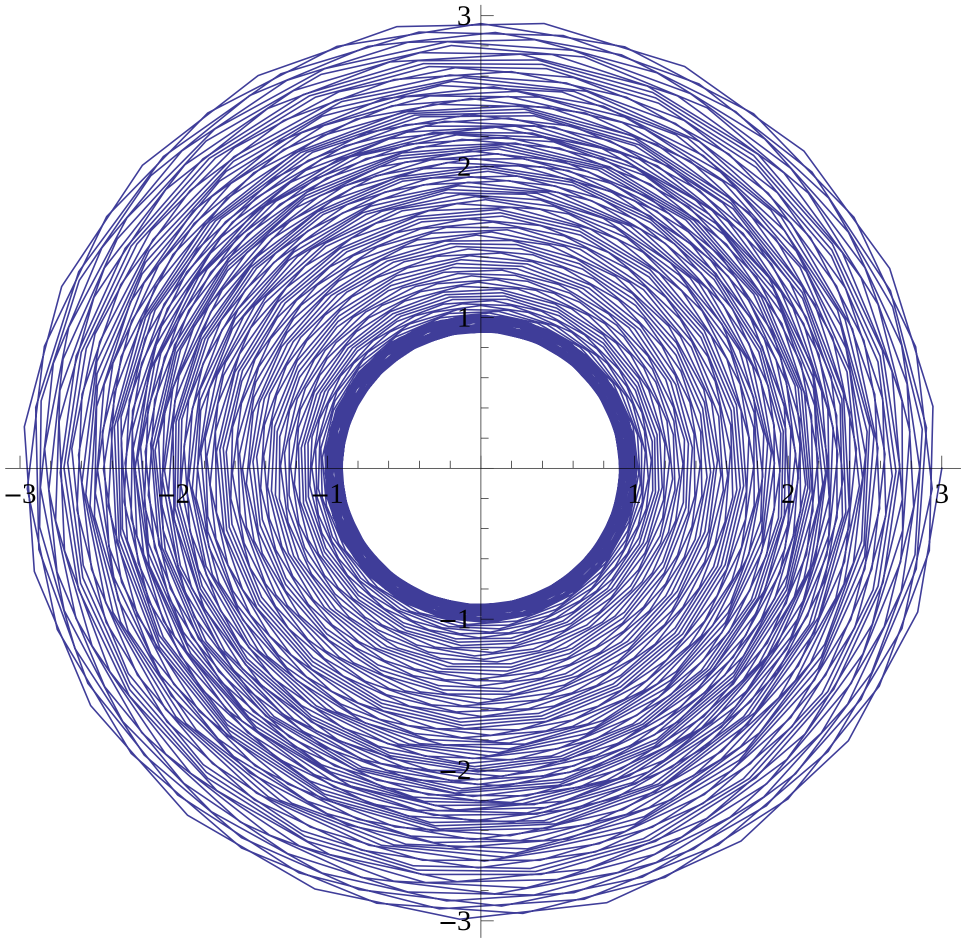
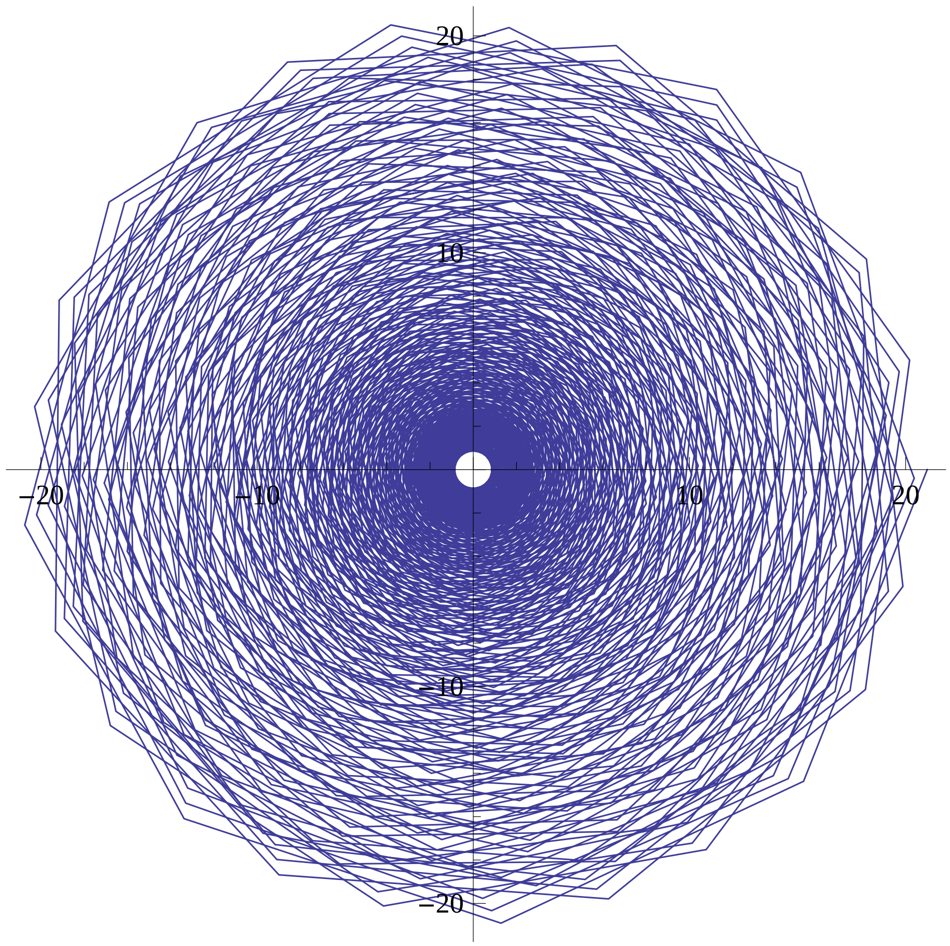
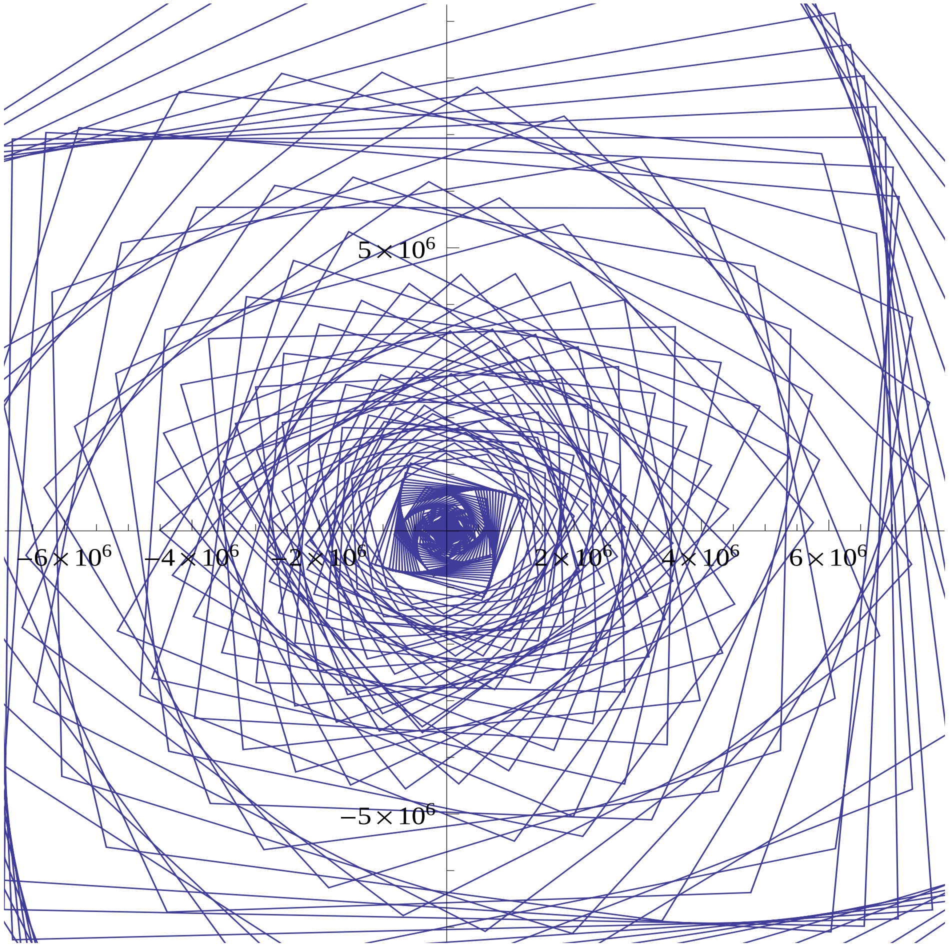


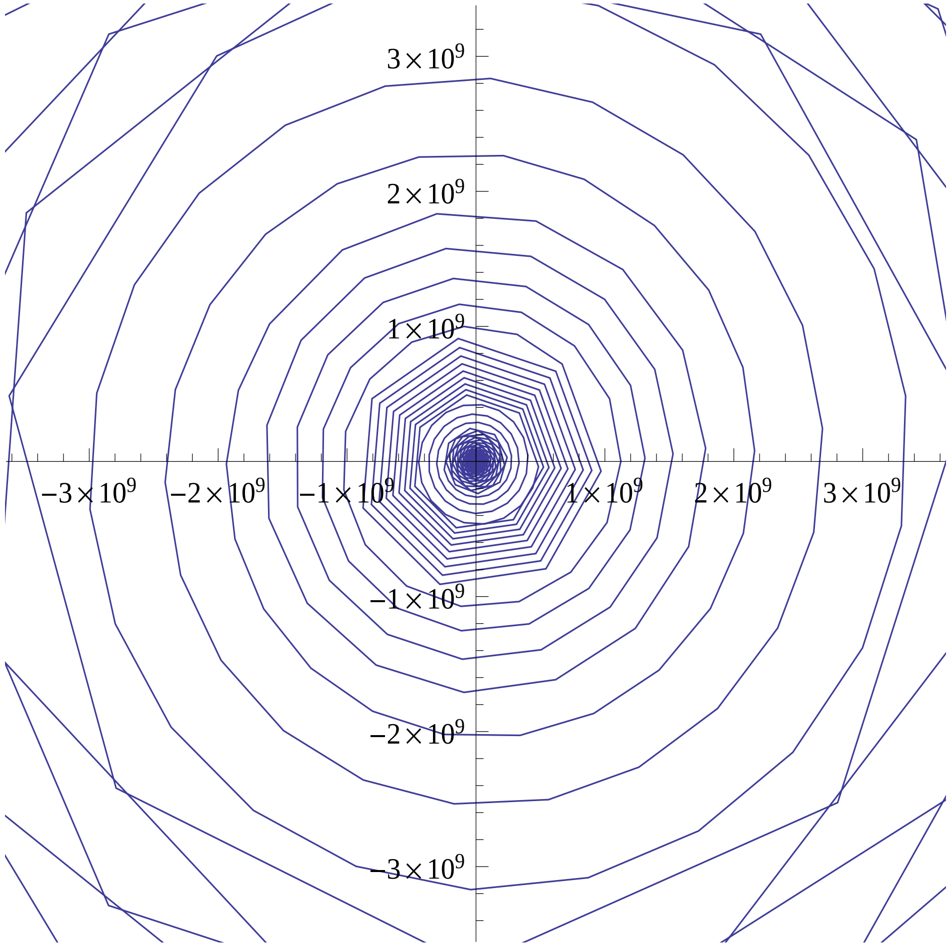
Complex Fibonacci spiral: t=8



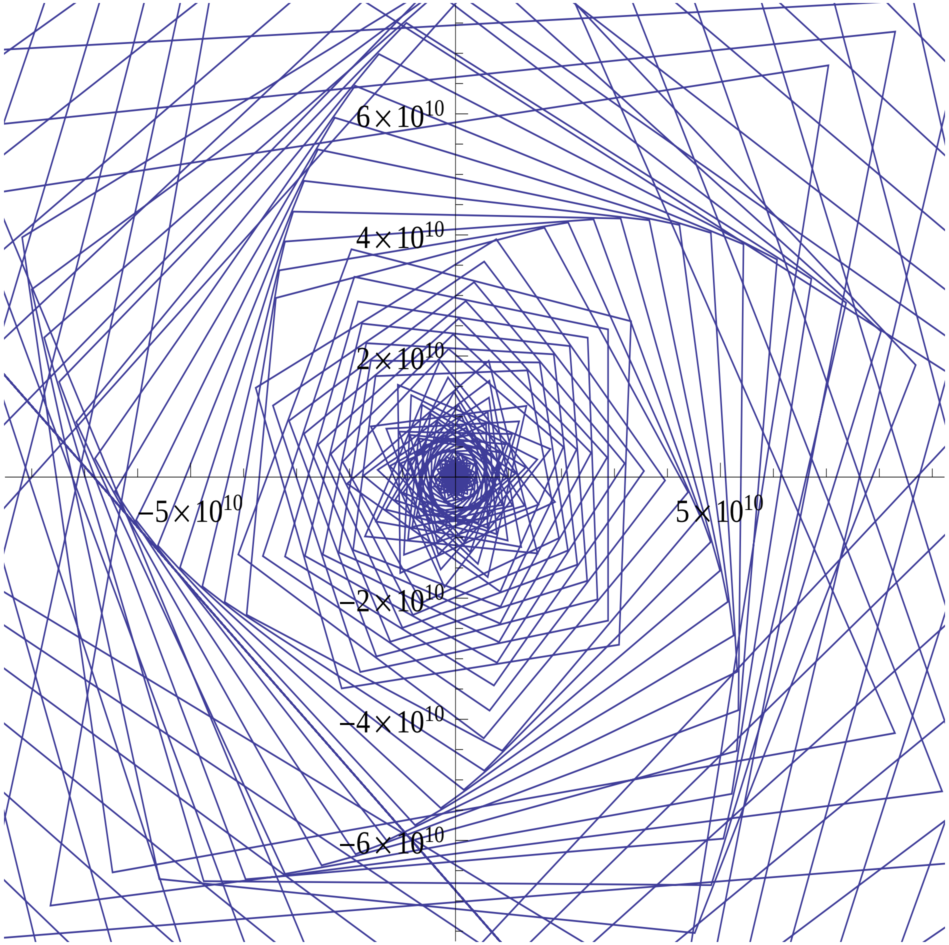
Complex Fibonacci spiral: t=50



Complex Fibonacci spiral: t= 70



Complex Fibonacci spiral: t=80



Two (N-) Qubit Plane

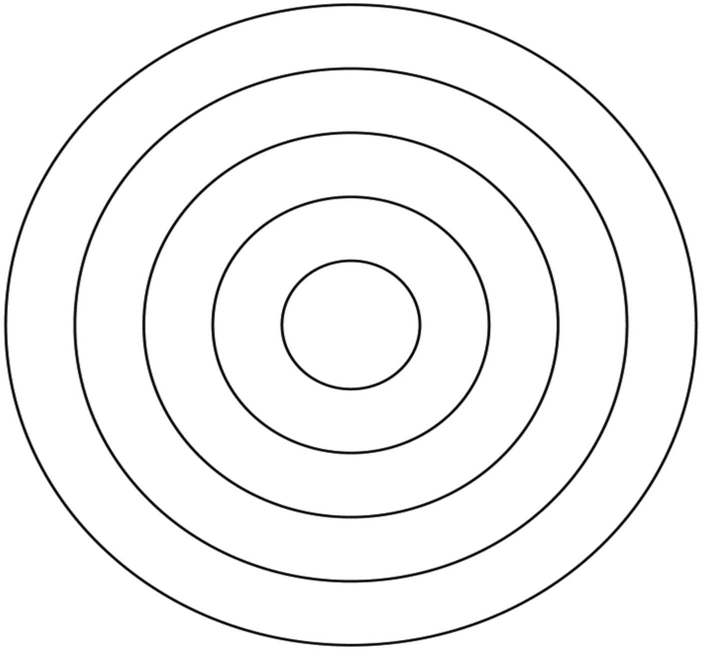
$$|z\rangle = \frac{|00\rangle + z|11\rangle}{\sqrt{1+|z|^2}}$$

Concurence $\rightarrow$ transition to symmetric state

$$C_z = |\langle \frac{1}{\bar{z}}|z\rangle| = \frac{2|z|}{1+|z|^2}$$

$z = 0 \rightarrow |00\rangle : C_0 = 0$
$z = \infty \rightarrow |11\rangle : C_\infty = 0$
$|z| = 1 \rightarrow$ maximally entangled state $C = 1$
It is invariant under rotation : $z \rightarrow ze^{i\alpha}$
reflections : $z \rightarrow \pm\bar{z}$ inversion : $z \rightarrow 1/\bar{z}$
Tuğçe Parlakgörür, O.K. Pashaev (2017)

Equi-entangled Circles



$C_r = \frac{2r}{1+r^2} = C_{1/r}$ equal under inversion

maximal $C = 1$ on unit circle $|z| = 1$

Apollonius Qubit States

$$z = \frac{w-1}{w+1}$$

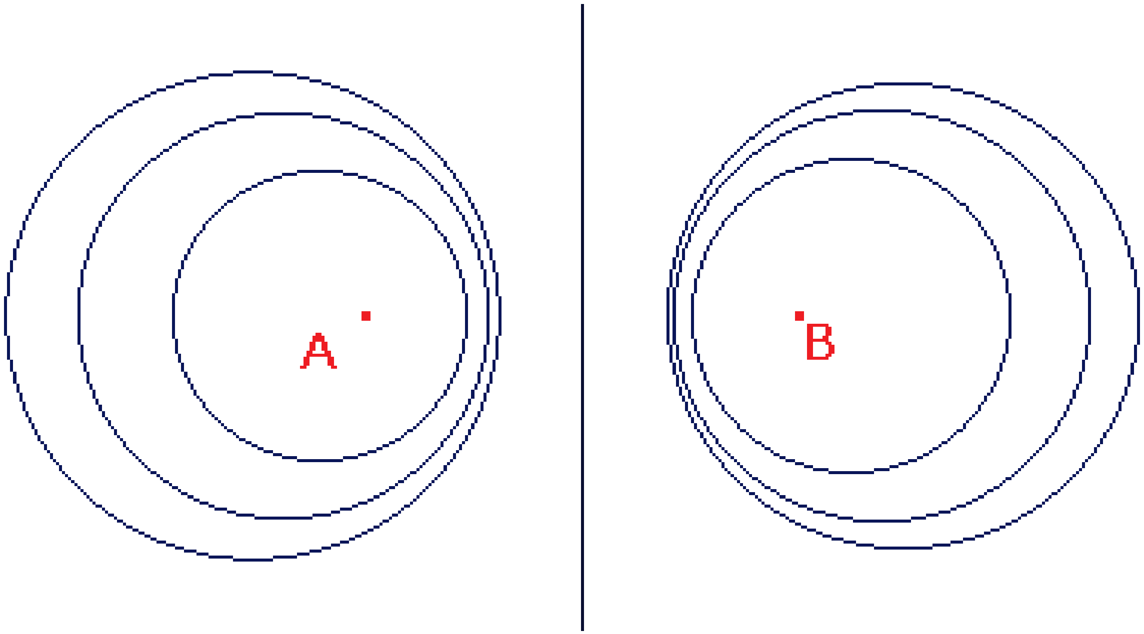
$$|w\rangle = \frac{(w+1)|00\rangle + (w-1)|11\rangle}{\sqrt{2(1+|w|^2)}}$$

$$C_w = \frac{|1-w^2|}{1+|w|^2}$$

Concurence $\rightarrow$ transition to symmetric states

$$C_w = |\langle -\bar{w}|w\rangle|$$

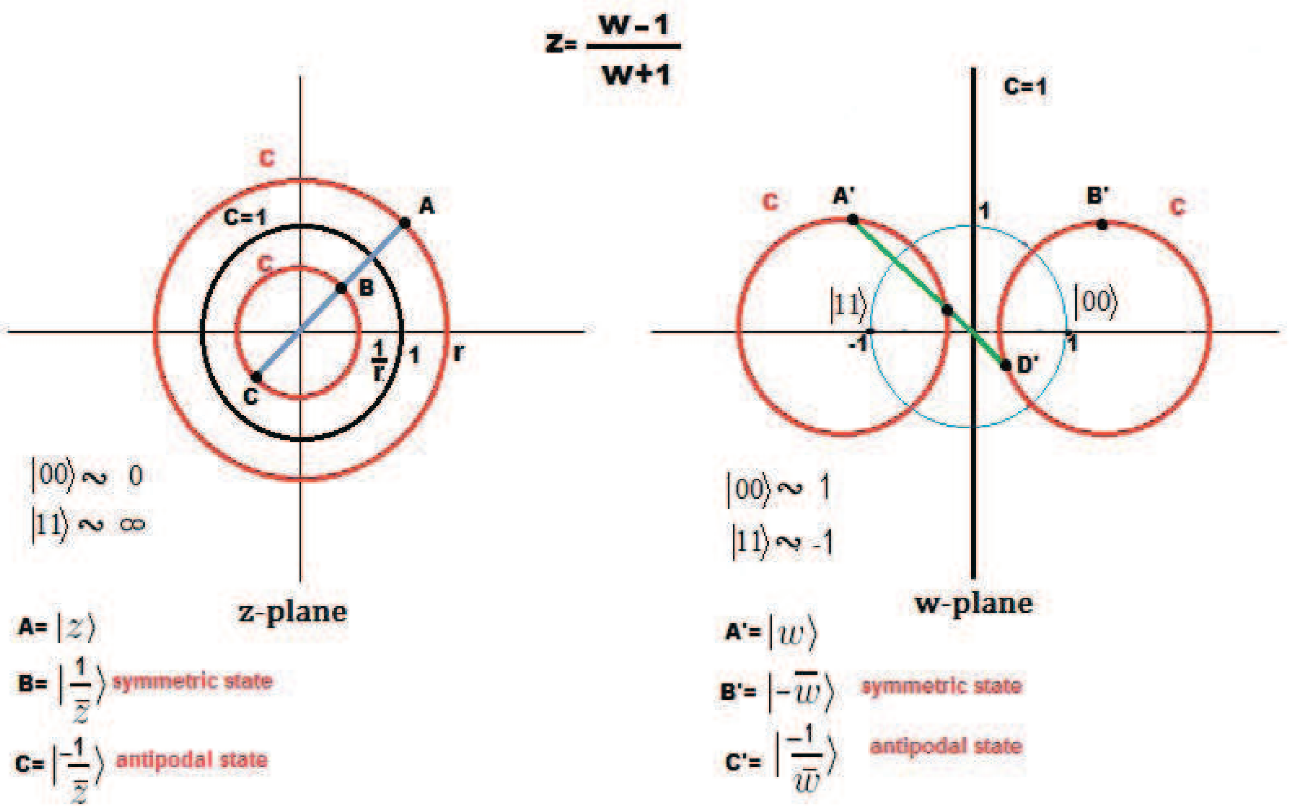
Apollonius Qubit States



$$C = 1$$

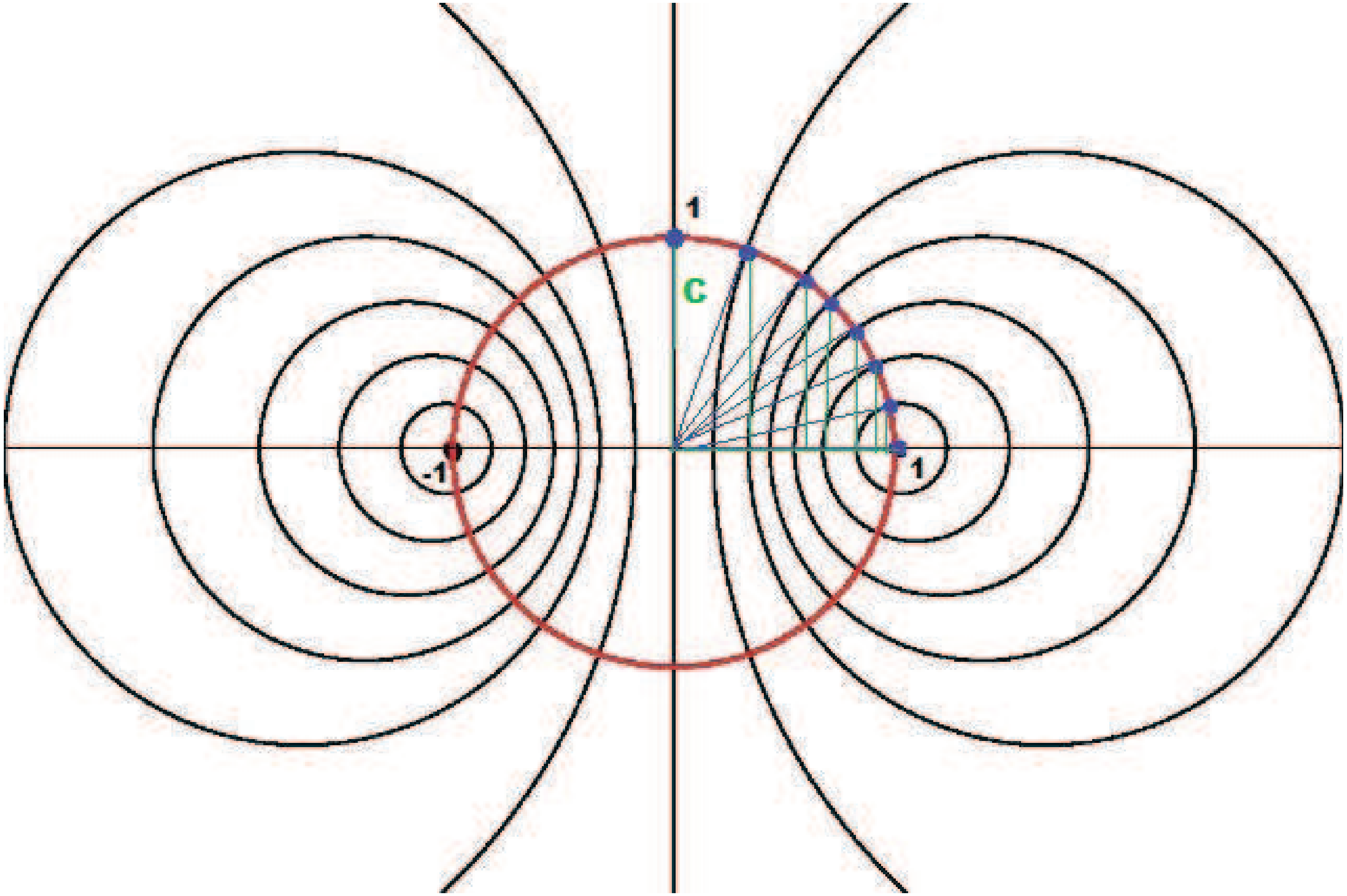
$C_A = 0, C_B = 0$ Electrostatic Analogy

Apollonius States



Apollonius States Concurence

$C = |\sin\alpha|$



T. Parlakgörür, Pashaev, 2017