Quantum Group Symmetry & Quantum Information for Kaleidoscope of Coherent States in Quantum Optics

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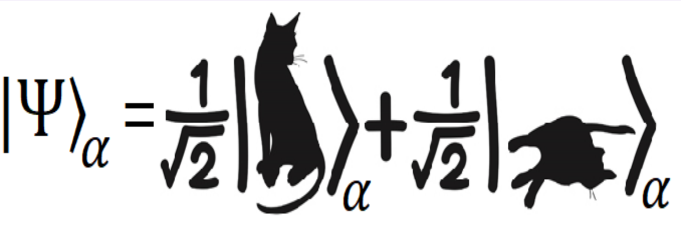
The Schrödinger cat states as superposition of Glauber's coherent states with opposite phases, become important tool for construction of qubits, as a units of quantum information in quantum optics. They correspond to even and odd quantum states.



Here we generalize this construction to the kaleidoscope of coherent states, related with regular n-polygon symmetry and the roots of unity. Superposition of coherent states with such symmetry plays the role of the quantum Fourier transform and provides the set of orthonormal quantum states, as a description of qutrits, ququats and qudits.

•https://www.youtube.com/watch?v=UjaAxUO6-Uw

Outline

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• The state $|\alpha\rangle$ obtained by applying displacement operator $D(\alpha)$ to the vacuum state is called Coherent State:

$$D(\alpha) = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}} \Rightarrow e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}}|0\rangle = |\alpha\rangle \ , \ \text{where} \ \left[\hat{a}, \hat{a}^\dagger\right] = \widehat{\mathrm{I}}$$

• $$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C}$$

Representation in the basis of Fock states :

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Representation of coherent states:

$$|\alpha\rangle = \frac{e^{\alpha\hat{a}^\dagger}}{\sqrt{e^{|\alpha|^2}}}|0\rangle$$

, which is instructive for our generalizations.

Heinsenberg Uncertainity relation: $\Delta\hat{q}\Delta\hat{p} \geq \frac{\hbar}{2}$

Coherent states are satisfying **minimum uncertainty** relation thus we say that they are "most classical states"

$$(\Delta\hat{q})_\alpha (\Delta\hat{p})_\alpha = \frac{\hbar}{2}$$

Inner Product of coherent states:

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\bar{\alpha}\beta}$$

$$\Rightarrow |\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = e^{-(|\alpha|^2+|\beta|^2-\overline{\alpha}\beta-\overline{\beta}\alpha)} = e^{-|\alpha-\beta|^2}$$

$\Rightarrow$ Coherent States are **NOT** orthogonal.

Outline

Schrödinger's Cat States:

$$|cat\rangle_e \sim |\alpha\rangle + |-\alpha\rangle \qquad |cat\rangle_o \sim |\alpha\rangle - |-\alpha\rangle$$

These cat states can be considered as a superposition of two states rotated by angle π and related with primitive root of unity: $q^4 = 1 \Rightarrow q^2 = -1 = e^{i\pi}$ and $1 + q^{2n} \equiv 1 + (-1)^n \equiv 2\delta_{n\equiv 0(mod2)}$

$$|0\rangle_\alpha = N_0(\,|\alpha\rangle + |q^2\alpha\rangle\,) \qquad |1\rangle_\alpha = N_1(\,|\alpha\rangle + q^2|q^2\alpha\rangle\,)$$

- $\langle\alpha|\alpha\rangle = \langle q^2\alpha|q^2\alpha\rangle = 1$
- $\langle\alpha|q^2\alpha\rangle = \langle q^2\alpha|\alpha\rangle = e^{-2|\alpha|^2}$

Normalizing $|0\rangle_\alpha$ by using inner products:

$$\boxed{|0\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle}{\sqrt{2}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2}}}}$$

Normalized Cat States:

$$e^{|\alpha|^2} + e^{q^2|\alpha|^2} = \sum_{n=0}^{\infty}\left(\frac{\left(|\alpha|^2\right)^n}{n!} + \frac{\left(q^2|\alpha|^2\right)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}(\underbrace{1+q^{2n}}_{2\delta_{n\equiv 0(mod2)}})$$

$$= 2\sum_{k=0}^{\infty}\frac{\left(|\alpha|^2\right)^{2k}}{(2k)!} = 2cosh(|\alpha|^2) = 2\,{}_0e^{|\alpha|^2}(mod2)$$

$$|0\rangle_{\alpha} = e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + |q^2\alpha\rangle}{\sqrt{2}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + |q^2\alpha\rangle}{2\sqrt{{}_0e^{|\alpha|^2}(mod2)}}$$

$$|1\rangle_{\alpha} = e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle}{\sqrt{2}\sqrt{e^{|\alpha|^2} + \bar{q}^2e^{q^2|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle}{2\sqrt{{}_1e^{|\alpha|^2}(mod2)}}$$

Hadamard Gate:

Matrix form:

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \end{bmatrix} = \mathbf{N}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & \bar{q}^2 \end{bmatrix}\begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \end{bmatrix}$$

, where $\mathbf{N} = \begin{bmatrix} N_0 & 0 \\ 0 & N_1 \end{bmatrix} = \frac{e^{\frac{1}{2}|\alpha|^2}}{\sqrt{2}}\begin{bmatrix} {}_0e^{|\alpha|^2} & 0 \\ 0 & {}_1e^{|\alpha|^2} \end{bmatrix}^{-\frac{1}{2}} (mod 2)$

Normalization factors is defined by even ($0 \; mod \; 2$) and odd ($1 \; mod \; 2$) exponential functions:

$${}_0e^{|\alpha|^2} \equiv \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{2k}}{(2k)!} = \frac{e^{|\alpha|^2} + e^{q^2|\alpha|^2}}{2} = \cosh|\alpha|^2 \, (mod \, 2)$$

$${}_1e^{|\alpha|^2} \equiv \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{2k+1}}{(2k+1)!} = \frac{e^{|\alpha|^2} + \bar{q}^2 e^{q^2|\alpha|^2}}{2} = \sinh|\alpha|^2 \, (mod \, 2)$$

Different representation of Schrödinger's Cat States:

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle \quad (mod\, 2) = \frac{\cosh\alpha\hat{a}^\dagger}{\sqrt{\cosh|\alpha|^2}}|0\rangle \quad ,$$

$$|1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle \quad (mod\, 2) = \frac{\sinh\alpha\hat{a}^\dagger}{\sqrt{\sinh|\alpha|^2}}|0\rangle.$$

Schrödinger cat states are eigenstates of $\hat{a}^2$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow \hat{a}^2|\mp\alpha\rangle = \alpha^2|\mp\alpha\rangle$$

$$\hat{a}^2|0\rangle_\alpha = \alpha^2|0\rangle_\alpha, \;\; \hat{a}^2|1\rangle_\alpha = \alpha^2|1\rangle_\alpha \Rightarrow \hat{a}^2|\psi\rangle_\alpha = \alpha^2|\psi\rangle_\alpha$$

Every eigenstate of $\hat{a}^2$ operator can be written as linear combination of orthonormal basis $\{|0\rangle_\alpha, |1\rangle_\alpha\}$ and they can be used to define the coherent qubit state:

$$\boxed{|\psi\rangle_\alpha = c_0|0\rangle_\alpha + c_1|1\rangle_\alpha \,, \text{ where } |c_0|^2 + |c_1|^2 = 1}$$

Annihilation operator $\hat{a}$ gives flipping between cat states :

$$\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_1}|1\rangle_\alpha \quad , \quad \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha \quad (mod\, 2)$$

Since $\widehat{N} = \hat{a}^\dagger\hat{a}$,we can easily calculate number of photons in Schrödinger's Cat States:

$$\begin{aligned} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha &= |\alpha|^2 \tanh|\alpha|^2, \\ {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha &= |\alpha|^2 \coth|\alpha|^2. \end{aligned}$$

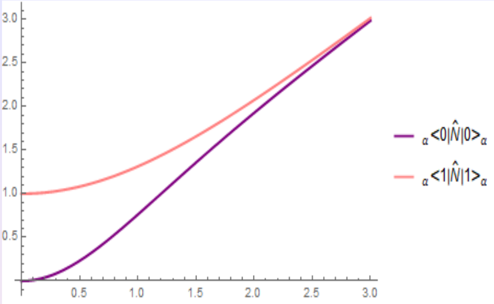

Figure : Number of photons in Schrödinger's Cat States

$$\lim_{|\alpha|\to\infty} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = \lim_{|\alpha|\to\infty} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = |\alpha|^2 = \langle \pm\alpha|\widehat{N}| \pm \alpha\rangle$$

Schrödinger's Kitten States :

$$\lim_{|\alpha|\to 0} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = 0 \quad \& \quad \lim_{|\alpha|\to 0} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = 1$$

Outline

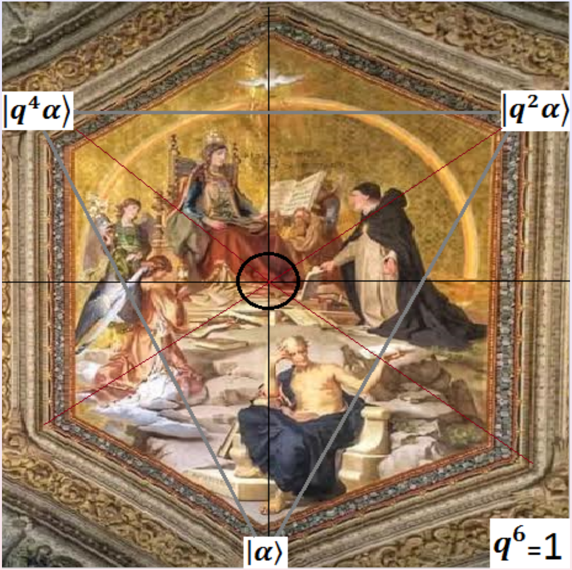

Figure : Trinity States

First, we construct orthonormal states as superposititon of $|\alpha\rangle, |q^2\alpha\rangle$ and $|q^4\alpha\rangle$ with $q^6 = 1$.

Define: $|0\rangle_\alpha = N_0 \left(|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle\right)$

$$|0\rangle_\alpha = N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (|n\rangle + q^{2n}|n\rangle + q^{4n}|n\rangle)$$

$$= N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ q^{2n} \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ q^{4n} \\ \vdots \end{bmatrix} \right)$$

Since we have equation $q^6 = 1$, we can write

$$q^{6n} = 1 \quad \Rightarrow \quad (q^{2n} - 1)(1 + q^{2n} + q^{4n}) = 0 \quad \Rightarrow$$

$$\boxed{1 + q^{2n} + q^{4n} = 3\delta_{n=0(mod3)}} (*)$$

$$|0\rangle_\alpha = 3N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{3k}}{\sqrt{(3k)!}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = 3N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{3k}}{\sqrt{(3k)!}} |3k\rangle$$

From Normalization of $|0\rangle_\alpha$:

$$_0e^{|\alpha|^2}(mod3) = \sum_{k=0}^{\infty} \frac{(|\alpha|^2)^{3k}}{(3k)!} = \frac{1}{3}\left(e^{|\alpha|^2} + e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2}\right)$$

Normalized Trinity states

$$|0\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{3\sqrt{{}_0e^{|\alpha|^2}(mod3)}}$$

$$|1\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle + \bar{q}^4|q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + \bar{q}^2 e^{q^2|\alpha|^2} + \bar{q}^4 e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle + \bar{q}^4|q^4\alpha\rangle}{3\sqrt{{}_1e^{|\alpha|^2}(mod3)}},$$

$$|2\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^4|q^2\alpha\rangle + \bar{q}^2|q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + \bar{q}^4 e^{q^2|\alpha|^2} + \bar{q}^2 e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^4|q^2\alpha\rangle + \bar{q}^2|q^4\alpha\rangle}{3\sqrt{{}_2e^{|\alpha|^2}(mod3)}}$$

General form of qutrit coherent state

$$\boxed{|\psi\rangle_\alpha = c_0|0\rangle_\alpha + c_1|1\rangle_\alpha + c_2|2\rangle_\alpha}$$

, where
$$\begin{aligned} |0\rangle_\alpha &= N_0\left(|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle\right) \\ |1\rangle_\alpha &= N_1\left(|\alpha\rangle + e^{-i\frac{2\Pi}{3}}|q^2\alpha\rangle + e^{i\frac{2\Pi}{3}}|q^4\alpha\rangle\right) \\ |2\rangle_\alpha &= N_2\left(|\alpha\rangle + e^{i\frac{2\Pi}{3}}|q^2\alpha\rangle + e^{-i\frac{2\Pi}{3}}|q^4\alpha\rangle\right) \end{aligned}$$

$$\& \quad |c_0|^2 + |c_1|^2 + |c_2|^2 = 1$$

Different form of of Trinity States

From $\hat{a}|0\rangle = 0$, we have relation between trinity states and $|0\rangle$:

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle, \quad |1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle, \quad |2\rangle_\alpha = \frac{{}_2e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_2e^{|\alpha|^2}}}|0\rangle$$

Matrix form:

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \end{bmatrix} = \mathbf{N} \underbrace{\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{q}^2 & \left(\bar{q}^2\right)^2 \\ 1 & \bar{q}^4 & \left(\bar{q}^4\right)^2 \end{bmatrix}}_{trinity\ \ gate} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} N_0 & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & N_2 \end{bmatrix} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{3}} \begin{bmatrix} {}_0e^{|\alpha|^2} & 0 & 0 \\ 0 & {}_1e^{|\alpha|^2} & 0 \\ 0 & 0 & {}_2e^{|\alpha|^2} \end{bmatrix}^{-1/2} \quad (mod\, 3)$$

$$\boxed{1 + \bar{q}^{2(n-k)} + \bar{q}^{4(n-k)} = 3\delta_{n \equiv k (mod 3)} \quad , 0 \le k \le 2}$$

Annihilation operator $\hat{a}$ gives flipping between trinity states:

$$\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_2}|2\rangle_\alpha, \quad \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha, \quad \hat{a}|2\rangle_\alpha = \alpha\frac{N_2}{N_1}|1\rangle_\alpha \quad (*)$$

We will use equations in $(*)$ to calculate number of photons in Trinity States;

$$_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2\left[\frac{{}_2e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{1+2e^{\frac{-3|\alpha|^2}{2}}cos\left(\frac{\sqrt{3}}{2}|\alpha|^2+\frac{2\Pi}{3}\right)}{1+2e^{\frac{-3|\alpha|^2}{2}}cos\left(\frac{\sqrt{3}}{2}|\alpha|^2\right)}\right]$$

$$_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = |\alpha|^2\left[\frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{1+2e^{\frac{-3|\alpha|^2}{2}}cos\left(\frac{\sqrt{3}}{2}|\alpha|^2\right)}{1+2e^{\frac{-3|\alpha|^2}{2}}cos\left(\frac{\sqrt{3}}{2}|\alpha|^2-\frac{2\Pi}{3}\right)}\right]$$

$$_\alpha\langle 2|\widehat{N}|2\rangle_\alpha = |\alpha|^2\left[\frac{{}_1e^{|\alpha|^2}}{{}_2e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{1+2e^{\frac{-3|\alpha|^2}{2}}cos\left(\frac{\sqrt{3}}{2}|\alpha|^2-\frac{2\Pi}{3}\right)}{1+2e^{\frac{-3|\alpha|^2}{2}}cos\left(\frac{\sqrt{3}}{2}|\alpha|^2+\frac{2\Pi}{3}\right)}\right]$$

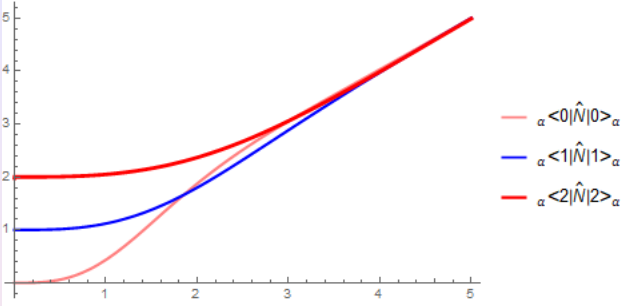

Figure : Photon numbers in Trinity States

$$\lim_{|\alpha|\to\infty} {}_\alpha\langle k|\widehat{N}|k\rangle_\alpha = |\alpha|^2 = \langle\alpha|\widehat{N}|\alpha\rangle \quad , \quad k = 0, 1, 2$$

Outline

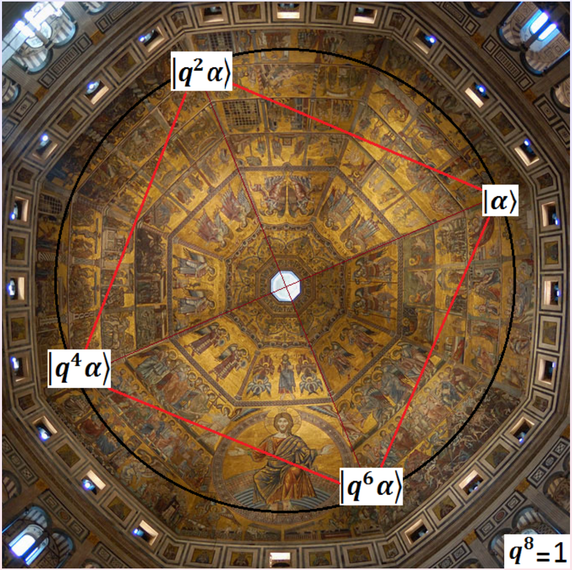

Figure : Quartet States

matrix form:

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \\ |3\rangle_\alpha \end{bmatrix} = \mathbf{N}\underbrace{\frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \overline{q}^2 & (\overline{q}^2)^2 & (\overline{q}^2)^3 \\ 1 & \overline{q}^4 & (\overline{q}^4)^2 & (\overline{q}^4)^3 \\ 1 & \overline{q}^6 & (\overline{q}^6)^2 & (\overline{q}^6)^3 \end{bmatrix}}_{Quartet\ gate}\begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \\ |q^6\alpha\rangle \end{bmatrix}$$

$$\mathbf{N} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{4}}\begin{bmatrix} {}_0e^{|\alpha|^2} & 0 & 0 & 0 \\ 0 & {}_1e^{|\alpha|^2} & 0 & 0 \\ 0 & 0 & {}_2e^{|\alpha|^2} & 0 \\ 0 & 0 & 0 & {}_3e^{|\alpha|^2} \end{bmatrix}^{-\frac{1}{2}}$$

$$\boxed{1 + \overline{q}^{2(n-k)} + \overline{q}^{4(n-k)} + \overline{q}^{6(n-k)} = 4\delta_{n\equiv k(mod4)} \quad , 0 \leq k \leq 3}$$

Number of Photons in Quartet States

We have flipping between states $|k\rangle_\alpha, \quad k = 0, 1, 2, 3$ with annihilation operator $\hat{a}$:

$$\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_3}|3\rangle_\alpha, \ \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha, \ \hat{a}|2\rangle_\alpha = \alpha\frac{N_2}{N_1}|1\rangle_\alpha, \ \hat{a}|3\rangle_\alpha = \alpha\frac{N_3}{N_2}|2\rangle_\alpha$$

Then,we can calculate number of photons in Quartet States;

$$\begin{aligned}
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha &= |\alpha|^2\left[\frac{{}_3e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{sinh|\alpha|^2 - sin|\alpha|^2}{cosh|\alpha|^2 + cos|\alpha|^2}\right] \\
{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha &= |\alpha|^2\left[\frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{cosh|\alpha|^2 + cos|\alpha|^2}{sinh|\alpha|^2 + sin|\alpha|^2}\right] \\
{}_\alpha\langle 2|\widehat{N}|2\rangle_\alpha &= |\alpha|^2\left[\frac{{}_1e^{|\alpha|^2}}{{}_2e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{sinh|\alpha|^2 + sin|\alpha|^2}{cosh|\alpha|^2 - cos|\alpha|^2}\right] \\
{}_\alpha\langle 3|\widehat{N}|3\rangle_\alpha &= |\alpha|^2\left[\frac{{}_2e^{|\alpha|^2}}{{}_3e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{cosh|\alpha|^2 - cos|\alpha|^2}{sinh|\alpha|^2 - sin|\alpha|^2}\right]
\end{aligned}$$

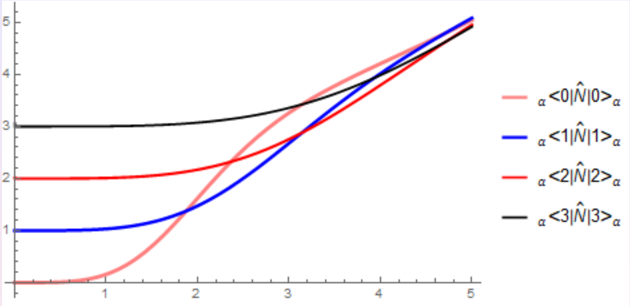

Figure : Photon numbers in Quartet States

Outline

Generalized n-Cat States

Consider the superposition of n coherent states which are belonging regular n-polygon and these states are rotated by angle $\frac{\pi}{n}$ which are related with primitive root of unity: $q^{2n} = 1$.

Inner product of q^{2k} coherent states:

- $\langle q^{2k}\alpha | q^{2k}\alpha\rangle = 1, \quad 0 \leq k \leq n-1$
- $\langle q^{2k}\alpha | q^{2l}\alpha\rangle = e^{|\alpha|^2(q^{2(l-k)}-1)}, \quad 0 \leq k, l \leq n-1$

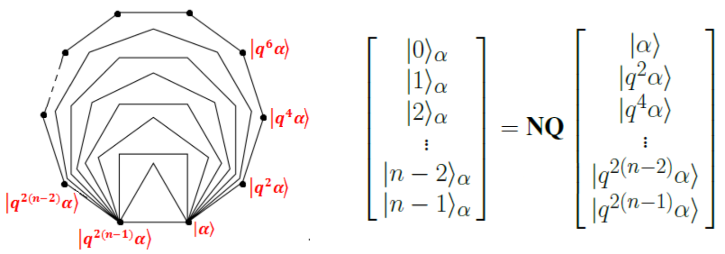

Figure : n-regular polygon states

$$|0\rangle_\alpha = N_0\left(|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle + ... + |q^{2(n-1)}\alpha\rangle\right)$$

$$= N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}(|n\rangle + q^{2n}|n\rangle + q^{4n}|n\rangle + ... + q^{2(n-1)}|n\rangle)$$

Lemma: For $q^{2n} = 1, 0 \leq s \leq n-1$,

- $1 + q^{2m} + q^{4m} + ... + q^{2m(n-1)} = n\delta_{m\equiv 0(modn)}$
- $1 + q^{2(m-s)} + q^{4(m-s)} + ... + q^{2(m-s)(n-1)} = n\delta_{m\equiv s(modn)}$

$$|0\rangle_\alpha = nN_0 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{nk}}{\sqrt{(nk)!}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = nN_0 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{nk}}{\sqrt{(nk)!}}|nk\rangle$$

Quantum Fourier Transformation

$$\begin{bmatrix} |\widetilde{0}\rangle_\alpha \\ |\widetilde{1}\rangle_\alpha \\ |\widetilde{2}\rangle_\alpha \\ |\widetilde{3}\rangle_\alpha \\ \vdots \\ |\widetilde{n-1}\rangle_\alpha \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \overline{q}^{\,2} & \overline{q}^{\,4} & \dots & \overline{q}^{\,2(n-1)} \\ 1 & \overline{q}^{\,4} & \overline{q}^{\,8} & \dots & \overline{q}^{\,4(n-1)} \\ 1 & \overline{q}^{\,6} & \overline{q}^{\,12} & \dots & \overline{q}^{\,6(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{q}^{\,2(n-1)} & \overline{q}^{\,4(n-1)} & \dots & \overline{q}^{\,2(n-1)^2} \end{bmatrix} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \\ |q^6\alpha\rangle \\ \vdots \\ |q^{2(n-1)}\alpha\rangle \end{bmatrix}$$

This matrix is Quantum Fourier transform where $\overline{q}^{\,2} = e^{\frac{-2\Pi i}{n}}$ is a n-th rooth of unity. The quantum fourier transform is unitary matrix which satisfy $QQ^\dagger = Q^\dagger Q = I$.

$$|\widetilde{k}\rangle_\alpha \longmapsto \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \overline{q}^{\,2jk} |q^{2j}\alpha\rangle \quad 0 \le k \le n-1$$

From these orthogonal states, we can construct normalized states which we called as "Kaleidoscope of Quantum coherent states":

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \\ \vdots \\ |n-1\rangle_\alpha \end{bmatrix} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{n}} \begin{bmatrix} {}_0e^{|\alpha|^2} & 0 & 0 & \dots & 0 \\ 0 & {}_1e^{|\alpha|^2} & 0 & \dots & 0 \\ 0 & 0 & {}_2e^{|\alpha|^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & {}_{(n-1)}e^{|\alpha|^2} \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} |\widetilde{0}\rangle_\alpha \\ |\widetilde{1}\rangle_\alpha \\ |\widetilde{2}\rangle_\alpha \\ \vdots \\ |\widetilde{n-1}\rangle_\alpha \end{bmatrix}$$

Define diagonal elements of normalization matrix:

$$N_s = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{n}} \left(f_s(|\alpha|^2)\right)^{\frac{-1}{2}} \quad \& \quad f_s(|\alpha|^2) =_s e^{|\alpha|^2}(mod\, n) = \sum_{k=0}^{\infty} \frac{(|\alpha|^2)^{nk+s}}{(nk+s)!}$$

Also, we can express them as superposition of standard exponentials;

$$ {}_se^{|\alpha|^2}(mod\, n) = \frac{1}{n} \sum_{k=0}^{n-1} \overline{q}^{2sk} e^{q^{2k}|\alpha|^2} \qquad , 0 \le s \le n-1$$

Derivative relation and Differential equation in (mod n):

$$\frac{\partial}{\partial|\alpha|^2}\left[{}_s e^{|\alpha|^2}\right] = {}_{s-1} e^{|\alpha|^2} \quad \& \quad \frac{\partial}{\partial|\alpha|^2}\left[{}_0 e^{|\alpha|^2}\right] = {}_{n-1}\, e^{|\alpha|^2},$$

$$\frac{\partial^n}{\partial(|\alpha|^2)^n}\left[{}_s e^{|\alpha|^2}\right] = {}_s\, e^{|\alpha|^2} = f_s(|\alpha|^2) \quad , \text{where } 0 \leq s \leq n-1$$

with proper initial values: $f_s^{(s)}(0) = 1$ and

$$f_s(0) = f_s'(0) = \ldots = f_s^{(s-1)}(0) = f_s^{(s+1)}(0) = \ldots = f_s^{(n-1)}(0) = 0.$$

Representation of Kaleidoscope of Quantum coherent states:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger}|0\rangle \;\Rightarrow\; |k\rangle_\alpha = \frac{{}_k e^{\alpha \hat{a}^\dagger}}{\sqrt{{}_k e^{|\alpha|^2}}}|0\rangle \quad (mod\ n)$$

Number of photons in **Kaleidoscope of Quantum coherent states**:

$$\hat{a}|s\rangle_\alpha = \alpha \frac{N_s}{N_{s-1}}|s-1\rangle_\alpha \Rightarrow {}_\alpha\langle s|\widehat{N}|s\rangle_\alpha = |\alpha|^2 \left[\frac{{}_{s-1}e^{|\alpha|^2}}{{}_{s}e^{|\alpha|^2}}\right], 0 < s < n-1$$

$$\& \qquad \hat{a}|0\rangle_\alpha = \alpha \frac{N_0}{N_{n-1}}|n-1\rangle_\alpha \Rightarrow {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2 \left[\frac{{}_{n-1}e^{|\alpha|^2}}{{}_{0}e^{|\alpha|^2}}\right]$$

$$\lim_{|\alpha|\to\infty} {}_\alpha\langle s|\widehat{N}|s\rangle_\alpha = |\alpha|^2 = \langle \pm\alpha|\widehat{N}| \pm \alpha\rangle$$

$$\lim_{|\alpha|\to 0} {}_\alpha\langle s|\widehat{N}|s\rangle_\alpha = s$$

Outline

Quantum Group Symmetry

Our kaleidoscope coherent states are eigenstates of operator $q^{2\hat{N}}$: $\boldsymbol{q^{2\hat{N}}|k\rangle_\alpha = q^{2k}|k\rangle_\alpha}$, $k = 0, 1..., n-1$.

$$\Sigma_3 \equiv q^{2\widehat{N}} = \hat{\mathrm{I}} \otimes \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q^2 & 0 & \dots & 0 \\ 0 & 0 & q^4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q^{2(n-1)} \end{pmatrix}, \quad \Sigma_1 \equiv \hat{\mathrm{I}} \otimes \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

These $n \times n$ matrices are called the Sylvester clock and shift matrices correspondingly satisfying $\boldsymbol{\Sigma_1 = (\hat{\mathrm{I}} \otimes Q)\Sigma_3(\hat{\mathrm{I}} \otimes Q^+)}$. They satisfy relations $\boldsymbol{\Sigma_1\Sigma_3 = q^2\Sigma_3\Sigma_1}$ and $\boldsymbol{\Sigma_1^n = \hat{\mathrm{I}}}$ & $\boldsymbol{\Sigma_3^n = \hat{\mathrm{I}}}$.

•**Quantum information** & **Coherent States:** Schrödinger Cat State can describe qubit as a unity of quantum information in binary system, Trinity states can be used in description of qutrit in ternary system. Generally, superposition of n-coherent states can describe qudit of quantum information for number with n-units.


Figure : Rotate the end and watch explosions of color inside the tube.